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E_8 Gauge Theory, and a Derivation of K-Theory from M-Theory

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The partition function of Ramond-Ramond p-form fields in Type IIA supergravity on a ten-manifold X contains subtle phase factors that are associated with T-duality, self-duality, and the relation of the RR fields to K-theory. The analogous partition function of M-theory on $X \times \mathbf{S}^1$ contains subtle phases that are similarly associated with E_8 gauge theory. We analyze the detailed phase factors on the two sides and show that they agree, thereby testing M-theory/Type IIA duality as well as the K-theory formalism in an interesting way. We also show that certain D-brane states wrapped on nontrivial homology cycles are actually unstable, that $(-1)^{F_L}$ symmetry in Type IIA superstring theory depends in general on a cancellation between a fermion anomaly and an anomaly of RR fields, and that Type IIA superstring theory with no wrapped branes is well-defined only on a spacetime with $W_7 = 0$.

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1. Introduction

The Ramond-Ramond (RR) p-form fields of Type II superstring theory are rather subtle objects, even in a limit in which they are treated as free fields. The subtlety arises in part from the fact that they are self-dual, and hence difficult to understand fully from a Lagrangian point of view, and in part from the fact that the D-branes that they couple to have a K-theory interpretation, and hence the RR fields themselves must be interpreted in K-theory. A framework for incorporating self-duality in the K-theory framework has been proposed [1] and used to resolve puzzles associated with global brane anomalies for Type IIA [2]. A unified approach to brane anomalies for both Type IIA and Type IIB in the K-theory framework has also been proposed [3].

In the present paper, we will focus primarily on the Type IIA theory. The RR partition function for Type IIA on a ten-manifold X can be written as a sum over the fluxes or periods of the RR fields, which are forms G_0, G_2, and G_4 of even order. (The G_{2p} of $p > 4$ are the electric-magnetic duals of these.) In the sum, subtle phase factors appear which are associated with the K-theory interpretation of RR fields, and thereby with $U(N)$ gauge theory.

One can also attempt to compare Type IIA on X to M-theory on $X \times \mathbf{S}^1$, or more generally on a circle bundle over X if $G_2 \neq 0$. In the M-theory framework, G_4 is identified with a component of the M-theory four-form, and G_2 with the first Chern class of the circle bundle. G_0 has no known M-theory origin. In the sum over periods of G_2 and G_4 in M-theory, there appear subtle phase factors. Locally, these arise from the familiar Chern-Simons interaction of eleven-dimensional supergravity, but to define the phase factors globally is a subtle story that involves use of E_8 gauge theory [4].

The purpose of the present paper is to compare the partition function of the RR fields as computed in Type IIA using self-duality and K-theory to the corresponding partition function computed in M-theory. We will find a nice match, which is a satisfying test of the M-theory/Type IIA duality and of the K-theory and E_8 gauge theory formalisms.

Throughout this paper, we keep the metric on X fixed. Moreover, for comparing to M-theory we take this metric large in string units. We perform quantum mechanics only for the RR fields and only in the free field approximation. Everything will come down to comparing the phases that appear on the two sides.

When the metric of X is scaled up by $g \to tg$, the action $\int d^{10}x\sqrt{g}|G_{2p}|^2$ for the Type IIA RR $2p$-form scales as t^{5-2p}. This implies that there is a sensible approximation of

keeping only G_4, whose periods have the smallest action, a sensible and better approximation of including G_4 and G_2, and finally one can do a complete calculation with G_0, G_2, and G_4 all included. To keep things relatively simple, while also (as it turns out) exhibiting most of the significant issues, we consider first the case of the contribution of G_4. We review the phase factor in the sum over G_4 flux in M-theory in section 2 and specialize to compactification on a circle in section 3. Then we introduce some necessary mathematical notions in section 4, and explain their role in Type II superstring theory in section 5. In section 6, we analyze the M-theory partition function with G_4 only, and in section 7 we demonstrate its agreement with the Type IIA partition function computed using self-duality and K-theory. In sections 8 and 9, we extend the analysis to include G_2, again showing complete agreement between the two sides. In section 10, we extend the Type IIA analysis to include G_0, but as G_0 has no known interpretation in M-theory, we are not able to compare the complete partition function to an M-theory result. In section 11, we explain a puzzle concerning the incorporation of the Neveu-Schwarz H-field and S-duality, and in section 12, we speculate about possible physical interpretation of some elements of the discussion.

Our notation is summarized in Appendix A. Appendix B contains details needed to complete one of the computations. Appendix C contains an outline of the Atiyah-Hirzebruch spectral sequence (AHSS) that compares K-theory to cohomology and underlies some of our considerations. Appendix D contains a technical argument for the existence of manifolds with nontrivial W_7.

2. Review of the Phase of the M-Theory Effective Action

We begin by recalling the origin in M-theory of the subtle phases that will be the focus of the present paper.

As predicted by Nahm's theorem [5], eleven-dimensional supergravity has a three-form field C, whose field strength we will denote as $G = dC$. Ever since the original construction of the classical theory [6], it has been known that the Lagrangian contains a Chern-Simons interaction, roughly

$$\int_Y C \wedge G \wedge G, \tag{2.1}$$

where Y is the M-theory spacetime.

M-theory has fivebranes, and G can have nonvanishing periods. Hence C is not globally defined as a three-form, and one must ask whether the Chern-Simons coupling

is globally defined. One might expect that, after determining the correct value of the quantum of G flux, the Chern-Simons interaction would be globally well-defined mod 2π. For this, it should be an integral multiple of

$$\frac{1}{(2\pi)^2}\int_Y C\wedge G\wedge G, \tag{2.2}$$

with C and G normalized so that the periods of G are integer multiples of 2π.

It turns out, however, that the Chern-Simons coupling of M-theory is smaller than this by a factor of 6 [4]. So how can the theory be consistent? First of all, there are a variety of gravitational corrections to the classical Chern-Simons interaction. There is a coupling $\int C\wedge I_8(R)$, where $I_8(R)$ is an eight-form constructed as a quartic polynomial in the Riemann tensor R. In addition, there is a gravitational correction to the quantization law of G, by virtue of which the periods of $G/2\pi$ are not in general integers. Rather, the general condition is that for any four-cycle U in spacetime,

$$\int_U \frac{G}{2\pi} = \frac{1}{2}\int_U \lambda \bmod \mathbf{Z}. \tag{2.3}$$

Here λ is defined as follows. The first Pontryagin class of a spin manifold is always divisible by 2, and there is a characteristic class λ (represented in de Rham cohomology by $\mathrm{tr}\, R\wedge R/16\pi^2$) such that $2\lambda = p_1(Y)$. Thus, the periods of $G/2\pi$ are integral or half-integral for cycles on which λ is even or odd.

There is, accordingly, an integral characteristic class, which we will call a, [1] that can be represented in de Rham cohomology as

$$G/2\pi = a - \lambda/2. \tag{2.4}$$

The full Chern-Simons coupling of M-theory is associated with the twelve-form

$$B_{12} = \frac{2\pi}{6}\left(a-\frac{\lambda}{2}\right)\left(\left(a-\frac{\lambda}{2}\right)^2 - \frac{1}{8}\left(p_2(Y)-\lambda^2\right)\right). \tag{2.5}$$

Here we have taken into account that $G/2\pi$ corresponds to $a-\lambda/2$. Thus, (2.5) corresponds to $(2\pi/6)(G/2\pi)^3$ plus corrections related to $CI_8(R)$.

In the usual fashion, one can try to define the Chern-Simons coupling in eleven dimensions by integrating B_{12} over a twelve-manifold. Indeed, by a result of Stong [7], there

[1] For an explanation of our notation see appendix A.

exists a twelve-dimensional spin manifold Z with boundary Y over which a extends. We interpret the Chern-Simons integral as a factor

$$\exp\left(i\int_Z B_{12}\right) \tag{2.6}$$

in the path integral. We must ask a key question: does this phase depend on the choice of Z? Such a dependence would not be physically acceptable, since the effective action should depend only on the spacetime Y, and not on the auxiliary choice of a twelve-manifold Z that is introduced for convenience in computation.

Before considering the Z-dependence, let us note that (2.6) is not quite the only problematic factor in the M-theory path integral at long distance. One must also worry about the Pfaffian (or square root of the determinant) of the Rarita-Schwinger operator D_{RS}. Though this operator is actually a more delicate construction, for our purposes (which mostly involve index theory of various sorts), D_{RS} is the Dirac operator $\not{D}$ coupled to $TY - 3\mathcal{O}$. Here TY is the tangent bundle to Y, and $\mathcal{O}$ is a trivial line bundle. The subtraction of $3\mathcal{O}$ accounts for the ghost fields required to fix the gauge invariances of the Rarita-Schwinger operator.[2] This Pfaffian, which we will write as $\mathrm{Pf}(D_{RS})$, is real but not positive definite; there is in general no natural definition of its sign. (Mathematically, $\mathrm{Pf}(D_{RS})$ is interpreted as a vector in a Pfaffian line rather than a real number.) The problematical factors in the M-theory effective action are thus the product

$$\mathrm{Pf}(D_{RS})\exp\left(i\int_Z B_{12}\right). \tag{2.7}$$

Although our interest in the present paper will mainly be on the second factor in (2.7), we pause to explain how to deal with the first factor. (This will enable us to fill in a gap in the discussion in [4], where the effective action was proved to be anomaly-free, but no absolute definition of its phase was given.) In the abstract there is no natural definition of the sign of $\mathrm{Pf}(D_{RS})$, but once a twelve-dimensional spin manifold Z of boundary Y is

[2] The tangent bundle TZ to Z splits near Y as $TZ = TY \oplus \mathcal{O}$, so by the Rarita-Schwinger operator in twelve dimensions, we will mean the Dirac operator coupled to $TZ - 4\mathcal{O}$. On the other hand, if $Y = \mathbf{S}^1 \times X$ or more generally if Y is a circle bundle over X, then TY can be replaced by $TX \oplus \mathcal{O}$, and hence in Type IIA, the Rarita-Schwinger operator (including the dilatino as well as the ghosts) is equivalent to a Dirac operator coupled to $TX - 2\mathcal{O}$.

chosen, there is a natural way to define the sign.[3] Let I_{RS} be the index of the Rarita-Schwinger operator on Z, computed with Atiyah-Patodi-Singer (APS) global boundary conditions [10]. This index is always even in 12 dimensions (or more generally in $8k+4$ dimensions for any k). We define the phase of $\mathrm{Pf}(D_{RS})$ to be $(-1)^{I_{RS}/2}$, or equivalently we define $\mathrm{Pf}(D_{RS})$ itself as

$$\mathrm{Pf}(D_{RS}) = (-1)^{I_{RS}/2} |\mathrm{Pf}(D_{RS})|, \tag{2.8}$$

where the absolute value $|\mathrm{Pf}(D_{RS})|$ can be defined by zeta function regularization and has no anomaly. The rationale for this definition is as follows. As the metric on Y is varied, a pair of eigenvalues of D_{RS} may pass through zero, in which case the sign of $\mathrm{Pf}(D_{RS})$ should jump. Precisely when this occurs, I_{RS} jumps by ± 2, so the right hand side of (2.8) changes sign precisely when the left hand side should. Hence, (2.8) gives a continuously varying (but Z-dependent) definition of $\mathrm{Pf}(D_{RS})$. If we use this definition, then the phase factor that must be considered in the M-theory path integral is

$$\Phi(Z) = (-1)^{I_{RS}/2} \exp\left(i \int_Z B_{12} \right). \tag{2.9}$$

Each factor here has been defined in a way that depends on Z. How can we prove that the product does not depend on Z? The key [4] is to use E_8 gauge theory. The role of E_8 does not fall completely out of the sky. If Y has a boundary, there is an E_8 vector supermultiplet propagating on the boundary. As shown in [11], anomaly cancellation along the boundary depends (among other things) on an anomaly inflow from the bulk, along the general lines considered in [12]. The anomaly inflow depends upon the fact that the Chern-Simons coupling is not gauge-invariant on an eleven-manifold with boundary. To cancel the anomalies, the key fact is that B_{12} has a relation to E_8 and Rarita-Schwinger index theory that we will now recall.

The homotopy groups $\pi_i(E_8)$ vanish for $1 \leq i \leq 14$ except for $i = 3$, where we have $\pi_3(E_8) = \mathbf{Z}$. Consequently, an E_8 bundle V on a twelve-manifold (or any manifold of dimension less than sixteen) is completely classified topologically by a four-dimensional characteristic class, which is represented in de Rham cohomology by $\mathrm{tr}\, F \wedge F/16\pi^2$. (Here F is the curvature of the bundle, and tr is $1/30$ times the trace in the adjoint representation.)

[3] The following observation is in the spirit of [8], and the idea was explained to us by D. Freed in commenting on [9], where a similar treatment was given for the heterotic string using the Dai-Freed theorem.

In particular, we can pick V so that its characteristic class is the class a associated with the C-field of M-theory. It is convenient if we can pick the connection on V so that the differential form $\mathrm{tr}\, F \wedge F/16\pi^2$ is precisely equal to the differential form $a = G/2\pi + \lambda/2$ that appears in the definition of B_{12}. But it is not really essential to be able to do this. For any choice of connection on V, these two four-forms will be equal for some C-field in its given topological class. If the phase of the M-theory effective action is well-defined (independent of the choice of Z) for some C-field in a given topological class, then it follows that this effective action is well-defined for every C-field in that class. For the change in the Chern-Simons coupling when C is continuously varied is given by a well-defined local integral over Y with no global subtleties.

Let I_{E_8} be the index of the Dirac operator D_{E_8} on Z, coupled to the E_8 bundle V, with APS global boundary conditions. Like I_{RS}, it is always even. On a twelve-manifold Z without boundary, I_{E_8} and I_{RS} are given by the index theorem in terms of the integrals over Z of certain twelve-forms i_{E_8} and i_{RS}. The crucial property of B_{12} is that it can be expressed in terms of these forms [11,4]:

$$\frac{B_{12}}{2\pi} = \frac{i_{E_8}}{2} + \frac{i_{RS}}{4}. \tag{2.10}$$

Hence, on a twelve-manifold Z without boundary, one has

$$\int_Z \frac{B_{12}}{2\pi} = \frac{I_{E_8}}{2} + \frac{I_{RS}}{4}. \tag{2.11}$$

Inserting this in (2.9), and using the fact that I_{E_8} and I_{RS} are even, one finds that $\Phi(Z)$ equals $+1$ for any closed twelve-manifold Z, independent of Z. Thus $\Phi(Z)$ is independent of Z if Z has no boundary.

It now follows, with a little more work, that $\Phi(Z)$ is independent of Z also if Z has a given boundary Y. The idea here is that if Z and Z' are twelve-dimensional spin-manifolds with the same boundary Y, and $\overline{Z} = Z \cup (-Z')$ is the closed twelve-manifold obtained by gluing Z and Z' along their boundary with a reversed orientation for Z', then [4]

$$\Phi(\overline{Z}) = \Phi(Z)\Phi(Z')^{-1}. \tag{2.12}$$

[4] The multiplicativity that leads to the following formula is clear for the second factor in the definition of $\Phi(Z)$. It also holds for the first factor, since $I_{RS}(\overline{Z}) = I_{RS}(Z) + I_{RS}(Z') = I_{RS}(Z) - I_{RS}(-Z')$. (Reversing the orientation of Z' changes the sign of the index: $I_{RS}(-Z') = -I_{RS}(Z')$.) This can be proved by picking on $\overline{Z}$ a metric that near the common boundary Y of Z and Z' has a long collar looking like $Y \times J$, where J is an interval in $\mathbf{R}$ very long compared to the radius of Y. We also assume on Y a generic metric such that D_{RS} has no zero modes on Y. Then a Rarita-Schwinger zero mode on $\overline{Z}$ converges (as J becomes very long) to a sum of zero modes on the two sides. This leads to the additivity of I_{RS}.

So, as $\Phi(\overline{Z}) = 1$, it follows that $\Phi(Z) = \Phi(Z')$, as promised.

This shows the well-definedness of the phase of the M-theory effective action. For our computations, however, it will generally be inconvenient actually to pick a Z with boundary Y. Instead, we will use an alternative expression that follows from the APS theorem for the index of the Dirac operator on a manifold with boundary.

Consider a Dirac operator D (such as D_{E_8} or D_{RS}) on the eleven-dimensional spin manifold Y. Let λ_i be its eigenvalues (which are real). Atiyah, Patodi, and Singer define the function

$$\eta(s) = \sum_{\lambda_i \neq 0} (\operatorname{sign} \lambda_i) |\lambda_i|^{-s}. \tag{2.13}$$

The sum converges for sufficiently large s and has an analytic continuation to $s = 0$. The value $\eta(0)$ is commonly called simply η. Let h be the number of zero modes of D. Now, suppose that Y is the boundary of a spin manifold Z (over which any data, such as an E_8 bundle, used in defining D are extended), and let $I(D)$ be the index of the extended operator D on Z, defined with APS global boundary conditions. Then the APS theorem (Theorem 4.2 in [10]) asserts that

$$I(D) = \int_Z i_D - \frac{h+\eta}{2}, \tag{2.14}$$

where i_D is the twelve-form whose integral on a closed twelve-manifold would equal $I(D)$.

In our case, we want a formula for $\int_Z B_{12}/2\pi$, which is related to index densities by (2.10). So from (2.14) we get:

$$\int_Z \frac{B_{12}}{2\pi} = \frac{1}{2} I_{E_8} + \frac{1}{4} I_{RS} + \frac{h_{E_8} + \eta_{E_8}}{4} + \frac{h_{RS} + \eta_{RS}}{8}. \tag{2.15}$$

If we insert this in the formula (2.9) for the phase Φ, then the I_{E_8} and I_{RS} terms can be dropped (using the fact that these indices are both even). We get that the phase of the effective action is

$$\Phi = \exp\left(2\pi i\left((h_{E_8} + \eta_{E_8})/4 + (h_{RS} + \eta_{RS})/8\right)\right). \tag{2.16}$$

This formula for the phase manifestly depends only on the fields on Y – not on an extension to Z. In general, one might expect that it would be hard to use, since the η invariant is a rather subtle thing. But it turns out that in the situations that we will encounter in the present paper (circle fibrations with the Neveu-Schwarz B-field set to

zero, i.e. with G pulled back from 10 dimensions), the expression (2.16) for the phase is very useful.

Let us verify using this formula that the integrand in the path integral is a smooth function of the metric on Y. In $8k+3$ dimensions, the eigenvalues of the Dirac operator coupled to a real vector bundle (such as the E_8 bundle V or the tangent bundle TY) have a two-fold degeneracy that comes from complex conjugation. As the metric on Y is varied, a pair of zero modes of D_{E_8} can pass through zero. Whenever an eigenvalue passes through zero, η jumps by ± 2. So the jumps in $h_{E_8} + \eta_{E_8}$ are by multiples of 4. Clearly this causes no discontinuity in Φ. On the other hand, $h_{RS} + \eta_{RS}$ likewise jumps by ± 4 when a pair of eigenvalues of D_{RS} pass through zero. When this occurs, Φ changes sign. We must remember, however, that the problematic factors in the path integral measure are not just Φ but the product $\Phi\,|\mathrm{Pf}(D_{RS})|$ of Φ with the absolute value of the Rarita-Schwinger Pfaffian. When a pair of eigenvalues of D_{RS} passes through zero, the Pfaffian $\mathrm{Pf}(D_{RS})$ changes sign, and its absolute value $|\mathrm{Pf}(D_{RS})|$ is continuous but not smooth. The sign change in Φ whenever $\mathrm{Pf}(D_{RS})$ has a zero of odd order is precisely right so that the product $\Phi\,|\mathrm{Pf}(D_{RS})|$ varies smoothly.

3. Reduction to Ten Dimensions and the Mod Two Index

As explained in the introduction, our initial goal will be to consider the case that $Y = X \times \mathbf{S}^1$, where X is a ten-dimensional spin manifold, and we use the supersymmetric (nonbounding) spin structure in the $\mathbf{S}^1$ direction. Moreover, we assume that the M-theory C-field is pulled back from a C-field on X. For this situation, we hope to compare M-theory on Y to Type IIA on X.

Note that $X \times \mathbf{S}^1$ with C a pullback from X has an orientation-reversing symmetry. Under this symmetry, the Chern-Simons coupling reverses sign, and the phase Φ is complex-conjugated. So in this situation, Φ must equal ± 1. As Φ takes values in a discrete set, it is a topological invariant in this situation. This is true for any value of the characteristic class $a \in H^4(X;\mathbf{Z})$ of the C-field.

Let us see how this works out in terms of (2.16). The Dirac operator changes sign under reflection of one coordinate, so the nonzero eigenvalues are in pairs $\lambda, -\lambda$. Hence the η-invariants are zero. So in this situation, the phase is just

$$\Phi = \exp\left(2\pi i (h_{E_8}/4 + h_{RS}/8)\right). \tag{3.1}$$

Let us analyze h_{E_8} and h_{RS}. A zero mode of the Dirac operator on $X \times \mathbf{S}^1$ must be constant in the $\mathbf{S}^1$ direction, so it is equivalent to a zero mode of the Dirac operator on X. Such zero modes may have either positive or negative chirality, and we have $h_{E_8} = h^+_{E_8} + h^-_{E_8}$, $h_{RS} = h^+_{RS} + h^-_{RS}$, where $h^\pm$ are the numbers of positive and negative chirality zero modes. Because complex conjugation reverses the chirality while mapping the real bundles V and TX to themselves, we have $h^+_{E_8} = h^-_{E_8}$, $h^+_{RS} = h^-_{RS}$. So we can write the phase as

$$\Phi = (-1)^{h^+_{E_8}} i^{h^+_{RS}}. \tag{3.2}$$

In $8k+2$ dimensions, the number of positive chirality zero modes of the Dirac operator with values in a real vector bundle is a topological invariant mod 2 [13]. Hence $h^+_{E_8}$ in particular is a topological invariant mod 2. We will denote this mod 2 invariant as $f(a)$.

Likewise, h^+_{RS} is a topological invariant mod 2. This is not enough to prevent jumping of the sign of Φ, but we must remember that whenever h^+_{RS} is nonzero, the other factor $|\mathrm{Pf}(D_{RS})|$, which multiplies Φ, is zero. If the Rarita-Schwinger mod 2 index is zero, then h^+_{RS} is generically zero and the phase reduces to

$$\Phi_a = (-1)^{f(a)}. \tag{3.3}$$

Even when the Rarita-Schwinger mod 2 index is nonzero, the a-dependence of the phase of the effective action, which is what we will primarily study in the present paper, is given by (3.3). (We will consider the effect of Rarita-Schwinger zero modes in section 3.3, where we compute the anomaly in $(-1)^{F_L}$.) Note that $f(a) = 0$ for $a = 0$, for in this case the E_8 bundle V is 248 copies of a trivial bundle, and has a vanishing mod 2 index.

Our next goal will be to discover useful properties of the E_8 mod 2 index $f(a)$.

3.1. Analysis Of The E_8 Mod 2 Index

As will gradually become clear, it is hopeless to find an elementary formula for $f(a)$. However, it is possible to find a relatively simple algebraic identity obeyed by $f(a)$, and to deduce what we need from this identity.

An analogy with Type IIA and K-theory may be useful. The Type IIA partition function can be expressed as a sum over RR fluxes, which are classified by $K(X)$. An important role in writing the partition function is played by a mod 2 invariant $j(x)$. For

$x \in K(X)$, $j(x)$ is defined as the mod 2 index with values in the KO (or real K-theory) class $x \otimes \overline{x}$. There is no elementary formula for $j(x)$, but there is a useful algebraic identity:

$$j(x+y) = j(x) + j(y) + \omega(x,y), \tag{3.4}$$

where here $\omega(x,y)$ is defined as the ordinary index of the Dirac operator with values in $x \otimes \overline{y}$. There is an elementary formula for this index (from the Atiyah-Singer index theorem) so (3.4) says that the difference $j(x+y) - j(x) - j(y)$ is a much more elementary invariant than j itself. In proving (3.4), one uses [1] the following property of the mod 2 index. Suppose that V is a real vector bundle whose complexification splits as $W \oplus \overline{W}$, where W is a complex vector bundle. Let $q(V)$ denote the mod 2 index with values in V and $I(W)$ the ordinary index with values in W. Then

$$q(V) = I(W) \bmod 2. \tag{3.5}$$

(It is also true that $q(V) = I(\overline{W}) \bmod 2$; indeed, in $8k+2$ dimensions, $I(\overline{W}) = -I(W)$.)

We need an analog of (3.4) for the E_8 mod 2 index. First we introduce an important concept. We will say that an element $a \in H^4(X;\mathbf{Z})$ can be "lifted to K-theory" if there exists, for some N, a rank N complex vector bundle E with $c_1(E) = 0$, $c_2(E) = -a$. The rationale behind this definition is that, in Type IIA superstring theory, RR fields are classified topologically by an element x of $K(X)$. The relation is

$$\frac{G}{2\pi} = \sqrt{\hat{A}}\, \mathrm{ch}\, x. \tag{3.6}$$

For $G_0 = G_2 = 0$, which is the case related to M-theory on $X \times \mathbf{S}^1$, this implies in particular

$$\begin{aligned} \frac{G_4}{2\pi} &= -c_2(x) \\ \frac{G_6}{2\pi} &= \frac{c_3(x)}{2}. \end{aligned} \tag{3.7}$$

Thus, when $G_0 = G_2 = 0$, $G_4/2\pi$ is in Type IIA always minus the second Chern class of a K-theory element. Consequently, a G-field in M-theory with characteristic class $-a$ has a straightforward interpretation in Type IIA only if a can be lifted to K-theory. We will eventually see that this restriction on a arises in M-theory in a more roundabout way.

If a is minus the second Chern class of an $SU(N)$ bundle for some N, we want a bound on how big N must be. To reduce the structure group of an $SU(N+1)$ bundle E to $SU(N)$, one needs to pick a section s of E that is everywhere nonzero. One can scale s

so $|s|^2 = 1$ everywhere, in which case s is a section of the bundle U of unit vectors in E. The fibers of U are copies of $\mathbf{S}^{2N+1}$. Over a base space of dimension k, the obstruction to finding a section of U is controlled by $\pi_i(\mathbf{S}^{2N+1})$ for $i < k$; these groups vanish if $2N+1 \geq k$.[5] So in particular, for $k = 10$, we can always reduce the structure group of an $SU(N)$ bundle to $SU(5)$.

Let a, a' be elements of $H^4(X;\mathbf{Z})$ that lift to K-theory. From what has just been said, we can assume that there are $SU(5)$ bundles E, E' with $c_2(E) = -a$, $c_2(E') = -a'$. We want to compute $f(a+a') - f(a) - f(a')$. From the two $SU(5)$ bundles E and E', we can, using the existence of an embedding $(SU(5)\times SU(5))/\mathbf{Z}_5 \subset E_8$, construct in a natural way an E_8 bundle whose characteristic class is $a + a'$. We simply embed E in the first $SU(5)$ and E' in the second. In the same way, replacing E or E' by a rank five trivial bundle, we get an E_8 bundle with characteristic class a or a'.

The decomposition of the E_8 Lie algebra under $SU(5) \times SU(5)$ is

$$\mathbf{248} = (\mathbf{24},\mathbf{1}) \oplus (\mathbf{1},\mathbf{24}) \oplus (\mathbf{5},\mathbf{10}) \oplus (\overline{\mathbf{5}},\overline{\mathbf{10}}) \oplus (\mathbf{10},\overline{\mathbf{5}}) \oplus (\overline{\mathbf{10}},\mathbf{5}). \tag{3.8}$$

Here $\mathbf{5}$ is the fundamental representation of $SU(5)$, $\mathbf{10}$ is its second antisymmetric power, and $\mathbf{24}$ is the adjoint representation. When we compute $f(a + a') - f(a) - f(a')$, the contributions of the $(\mathbf{24},\mathbf{1})$ and $(\mathbf{1},\mathbf{24})$ cancel out. The mod 2 index with values in $(\mathbf{5},\mathbf{10}) \oplus (\overline{\mathbf{5}},\overline{\mathbf{10}})$ is, from (3.5), the mod 2 reduction of the ordinary index of the $(\mathbf{5},\mathbf{10})$, and likewise the mod 2 index with values in $(\mathbf{10},\overline{\mathbf{5}}) \oplus (\overline{\mathbf{10}},\mathbf{5})$ is the mod 2 reduction of the ordinary index with values in $(\mathbf{10},\overline{\mathbf{5}})$.

Let $\wedge^2 E$, $\wedge^2 E'$ denote the bundles associated to E and E', respectively, in the $\mathbf{10}$ of $SU(5)$. We need to compute the ordinary index with values in $E \otimes \wedge^2 E' \oplus \wedge^2 E \otimes \overline{E}'$. To compute $f(a + a') - f(a) - f(a')$, we only want terms in the index formula that involve Chern classes of both E and E'. In general, in ten dimensions, the terms in the index formula for $A \otimes B$ that involve Chern classes of both A and B are (if $c_1(A) = c_1(B) = 0$)

$$-\int_X \frac{c_2(A)c_3(B) + c_3(A)c_2(B)}{2}. \tag{3.9}$$

We have

$$c_2(\wedge^2 E) = 3c_2(E),\ c_3(\wedge^2 E) = c_3(E), \tag{3.10}$$

[5] The reader may want to consult [14] for an introduction to obstruction theory for physicists.

and similarly for E'. Using (3.9) and (3.10), the contribution to the index of $E\otimes\wedge^2E'\oplus\wedge^2E\otimes\overline{E}'$ that survives in $f(a+a')-f(a)-f(a')$ is

$$\int_X c_2(E)c_3(E') \bmod 2. \tag{3.11}$$

At first sight, there is something perplexing about this result. We have partly characterized E and E' by requiring that $c_1(E)=c_1(E')=0$, $c_2(E)=-a$, $c_2(E')=-a'$, but we have said nothing about $c_3(E)$ and $c_3(E')$. They are not uniquely determined by the values of c_1 and c_2. However, for (3.11) to make sense, it must be that $c_3(E')$ is uniquely determined mod 2 by the conditions on $c_1(E')$ and $c_2(E')$. Moreover, it must be that $c_3(E)c_2(E')=c_2(E)c_3(E')$ mod 2.

To explain these points, we begin with the following fact. Let F be a complex vector bundle on $\mathbf{S}^6$. Then from the index theorem, the index $I(F)$ of the Dirac operator on $\mathbf{S}^6$ with values in F is

$$I(F)=\int_{\mathbf{S}^6}\frac{c_3(F)}{2}. \tag{3.12}$$

In particular, it follows that for a bundle F on $\mathbf{S}^6$, $c_3(F)$ is always divisible by 2 – congruent to 0 mod 2.

In general, for a rank N complex vector bundle F on an arbitrary manifold X, $c_3(F)$ is not necessarily congruent to 0 mod 2, but it is determined mod 2 in terms of $c_1(F)$ and $c_2(F)$. This can be deduced from (3.12) if one starts with a triangulation of X and inductively constructs the bundle F on the p-skeleton for $p=0,1,2,\ldots$.[6] Suppose that F has been defined on the $(p-1)$-skeleton of X and that one wishes to define it on the p-skeleton. Consider a particular p-simplex over which one wishes to extend F. Topologically, it is a p-dimensional ball $\mathbf{B}^p$ whose boundary is a sphere $\mathbf{S}^{p-1}$. F must be trivial on $\mathbf{S}^{p-1}$ or no extension over $\mathbf{B}^p$ exists. If F is trivial on the boundary, F can be extended over $\mathbf{B}^p$ but the extension may not be unique: given any one extension, the others can be obtained from it by "twisting" by an arbitrary element of $\pi_{p-1}(U(N))$, which (for large enough N) is nonzero precisely if p is even. (The twist in the bundle is made by cutting out a small p-ball from $\mathbf{B}^p$ and gluing it back in while making a gauge transformation on the boundary by an element of $\pi_{p-1}(U(N))$.) The twist shifts $c_{p/2}(F)$ on the simplex in question by an amount equal to $c_{p/2}(F)$ for some $U(N)$ bundle on $\mathbf{S}^p$ (namely, the bundle made by cutting and gluing using the same element of $\pi_{p-1}(U(N))$).

[6] Here again the reader may consult [14] for background on obstruction theory.

This is the full indeterminacy in $c_{p/2}(F)$ once F is given on the $(p-1)$-skeleton. So in particular, setting $p=6$ and using the fact that on $\mathbf{S}^6$, $c_3(F)$ is always even, it follows that $c_3(F)$ is determined mod 2 in terms of $c_1(F)$ and $c_2(F)$, which completely determine F on the five-skeleton.

When $c_1(F)=0$, the relation is $c_3(F)=Sq^2(c_2(F))$ mod 2, where Sq^2 is a certain cohomology operation known as a Steenrod square. We will give an introduction of sorts to Steenrod squares in section 4.1 and explain this formula. For the moment the reader can simply think of Sq^2 as a mysterious linear map from degree 4 to degree 6 cohomology. Setting $c_3(E')=Sq^2(E')$ mod 2, we can restate the result in (3.11) as follows:

$$f(a+a')=f(a)+f(a')+\int_X a\cup Sq^2a'. \tag{3.13}$$

A standard property of the Steenrod squares, explained in section 4.1 below, is that on a spin manifold,

$$\int_X a\cup Sq^2a'=\int_X Sq^2a\cup a'. \tag{3.14}$$

So the right hand side of (3.13) is symmetric in a and a', as it must be.

3.2. The role of cobordism theory

Our next goal is to show that (3.13) actually holds for arbitrary $a,a'\in H^4(X;\mathbf{Z})$, whether or not they can be lifted to K-theory. It follows from general considerations that $f(a)$ must be quadratic, simply because we are working in a dimension – 10 – which is less than three times four (where four is the degree of the class a). In other words, the most elementary cubic function of a would be $\int a\cup a\cup a$, which can be nonzero on a manifold of dimension $12=3\times 4$. All less elementary cubic functions appear in dimensions still higher than 12, roughly because all cohomology operations raise the degree of a cohomology class. (As we explain in section 4, the Steenrod squares are examples of cohomology operations raising the degree of a class.) Functions of higher order than cubic require a yet higher dimension. Nevertheless, this argument leaves open the possibility that there is a relation of the general form of (3.13) with some more general bilinear function on the right hand side. We will show this is not the case. For this, we will use a rather abstract argument based on cobordism theory. The reader may wish to accept (3.13) for all a,a' and skip this subsection. On the other hand, we have found the techniques described below to be a powerful tool in analyzing this and several related problems.

The mod 2 index of the Dirac operator coupled to a bundle V on a spin manifold vanishes if X is the boundary of a spin manifold over which V extends. So in particular, our E_8 mod 2 index $f(a)$ vanishes if X is the boundary of an eleven-dimensional spin manifold Y over which a extends. Similarly, consider the quantity

$$Q(a_1, a_2) = f(a_1 + a_2) - f(a_1) - f(a_2). \tag{3.15}$$

It is a function of two degree four classes a_1, a_2, and vanishes if both a_i can be extended to an eleven-dimensional spin manifold bounding X. Invariants such as $f(a)$ and $Q(a_1, a_2)$ are related to a generalized (co)homology theory known as (co)bordism theory. We now briefly explain these terms.

Recall that an n-manifold M_n is cobordant to zero if there exists an $(n+1)$-manifold B_{n+1} such that $\partial B_{n+1} = M_n$. We can ask that B_{n+1} carries structures carried by M, and thus we can define, for example, the spin bordism groups Ω_n^{spin}, where M_n is spin and B_{n+1} is required to be spin. An element of Ω_n^{spin} is a spin manifold, which is considered to be zero if it is the boundary of a spin manifold. To define the group structure of Ω_n^{spin}, manifolds (representing elements of Ω_n^{spin}) are added by taking their disjoint union.

Now let X be a topological space. We can define the "bordism groups of X" by taking sets of pairs (M_n, μ) where M_n is equipped with a continuous map $\mu : M_n \to X$. The equivalence relation on pairs is defined by declaring that (M_n, μ) is cobordant to zero if there is a bounding manifold B_{n+1} *together with* an extension of μ, i.e. $\widetilde{\mu} : B_{n+1} \to X$. The resulting bordism group is denoted by $\Omega_n(X)$. If M_n carries a structure we can then require that B_{n+1} also carry this structure. For example if M_n, B_{n+1} are required to be spin we define $\Omega_n^{spin}(X)$. The map $X \to \Omega_n(X)$ defines a generalized homology theory. The spin bordism groups Ω_n^{spin} with no X specified can be regarded as $\Omega_n^{spin}(X)$ for X a point. $\widetilde{\Omega}_n(X)$ is defined as the kernel of the natural map from $\Omega_n(X)$ to Ω_n in which one "forgets" X (or maps X to a point). A class (M_n, μ) in $\Omega_n(X)$ represents an element of $\widetilde{\Omega}_n(X)$ if (forgetting μ) M_n is a boundary.

Let us now interpret $f(a)$ in cobordism theory. It is a $\mathbf{Z}_2$-valued function $f(a)$ of a single cohomology class a, which vanishes when a extends to a spin manifold B bounding X. $f(a)$ is (therefore) additive on disjoint unions. To give a degree four class a on X is to give a map $\mu : X \to K(\mathbf{Z}, 4)$ to the universal space $K(\mathbf{Z}, 4)$ that classifies four-dimensional cohomology. The class $a \in H^4(X; \mathbf{Z})$ can be extended to a bounding manifold B – ensuring that $f(a) = 0$ – if and only if the map μ can be extended to a map from

B to $K(\mathbf{Z},4)$. In this case, the pair X,a is zero as an element of $\Omega_{10}^{spin}(K(\mathbf{Z},4))$. Thus, $f(a)$ can be regarded as a $\mathbf{Z}_2$-valued invariant of $\Omega_{10}^{spin}(K(\mathbf{Z},4))$ or more precisely as an element of $\mathrm{Hom}(\Omega_{10}^{spin}(K(\mathbf{Z},4)),\mathbf{Z}_2)$. Moreover, since $f(0)=0$, f can be viewed as an element of $\mathrm{Hom}(\widetilde{\Omega}_{10}^{spin}(K(\mathbf{Z},4),\mathbf{Z}_2)$. Replacing Ω by $\widetilde{\Omega}$ leads to important simplifications in the computation described below.

The quotient group $\widetilde{\Omega}_{10}^{spin}(K(\mathbf{Z},4))$ has been computed by Stong [7] and shown to be $\mathbf{Z}_2\times\mathbf{Z}_2$. Thus, there are two independent $\mathbf{Z}_2$-valued invariants of the pair (X,a). One such invariant is elementary:

$$v(a)=\int a\cup w_6=\int a\cup Sq^2\lambda \tag{3.16}$$

(here w_i are the Stiefel-Whitney class of X; the two formulas are equivalent because on a spin manifold $w_6=Sq^2\lambda$). In particular, $v(a)$ is a linear function of a: $v(a+a')=v(a)+v(a')$. Our invariant $f(a)$ is not linear, as we see from (3.13), so it is the "second" invariant of $\widetilde{\Omega}^{10}(K(\mathbf{Z},4))$.

Now that we have put $f(a)$ in the context of cobordism theory, let us turn to the bilinear identity for f. The object $Q(a_1,a_2)=f(a_1+a_2)-f(a_1)-f(a_2)$ is a homomorphism from $\Omega_{10}(K(\mathbf{Z},4)\times K(\mathbf{Z},4))$ to $\mathbf{Z}_2$. Moreover, Q vanishes if either a_1 or a_2 is zero. This means that we can replace $K(\mathbf{Z},4)\times K(\mathbf{Z},4)$ by $K(\mathbf{Z},4)\wedge K(\mathbf{Z},4)$, where $\wedge$ denotes the smashed product of two spaces

$$X\wedge Y=X\times Y/(X\vee Y). \tag{3.17}$$

($X\wedge Y$ is obtained from $X\times Y$ by picking points $p\in X$ and $q\in Y$, and collapsing $p\times Y\cup Z\times q$ to a point. An arbitrarily selected point is often denoted $\{*\}$.) It is often inconvenient to work directly with the smash product $\Omega(X\wedge Y)$ and technically more convenient to work with relative bordism groups $\Omega(X\times Y,X\vee Y)$. In general, relative bordism groups $\Omega_n(X,A)$ are defined as above by allowing M_n to have a nonempty boundary and considering maps of pairs $f(M_n,\partial M_n)\to(X,A)$; in other words, f maps the boundary of M_n to A. There is a natural notion of when such a pair should be considered cobordant to zero.

Now, putting the above remarks together and taking into account that we also want spin manifolds, our quantity $Q(a_1,a_2)$ in (3.15) is a homomorphism to $\mathbf{Z}_2$ of the relative bordism group

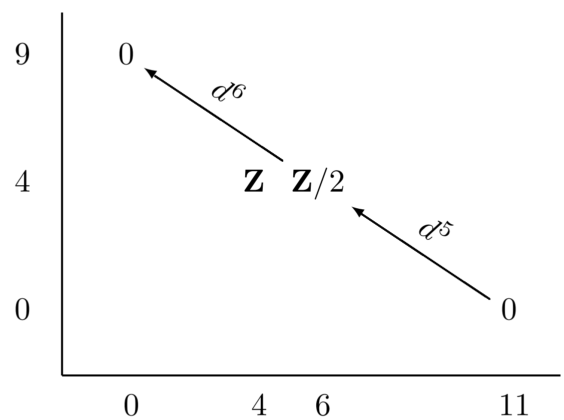
$$\Omega_{10}^{spin}(K(\mathbf{Z},4)\times K(\mathbf{Z},4),K(\mathbf{Z},4)\times\{*\}\cup\{*\}\times K(\mathbf{Z},4)). \tag{3.18}$$

We would therefore like to compute the group (3.18).

The computation of groups in "generalized (co)homology theories" is greatly facilitated by the "Atiyah-Hirzebruch spectral sequence." We will explain in greater depth how this works for K-theory in appendix C. Here we simply note that we can regard $K(\mathbf{Z},4) \times K(\mathbf{Z},4) \to K(\mathbf{Z},4)$ (the projection map to the second factor) as a (rather trivial) fibration, and apply [15] remark 2, pg 351. See also [16], ch. 1 sec. B. This allows us to construct the groups from a spectral sequence whose E^2 term is

$$E^2_{p,q} = \widetilde{H}_p(K(\mathbf{Z},4); \widetilde{\Omega}^{spin}_q(K(\mathbf{Z},4))) \tag{3.19}$$

Now the differentials act as $d^r : E^r_{p,q} \to E^r_{p-r,q+r-1}$ and thus change total degree by one. Thus we are interested in the above groups for $p+q = 9, 10$ (to compute kernels) and $p+q = 10, 11$ to compute images. At this point a very lucky fact occurs: the groups $\widetilde{H}_p(K(\mathbf{Z},4); G)$ for $G = \mathbf{Z}, \mathbf{Z}_2$ and $\widetilde{\Omega}^{spin}_q(K(\mathbf{Z},4)))$ are very sparse in low dimensions! Indeed, from Eilenberg-MacLane [17] we get $\widetilde{H}_i(K(\mathbf{Z},4); \mathbf{Z}) = 0$ for $i = 0,1,2,3,5,7$ while $\widetilde{H}_4(K(Z,4); \mathbf{Z}) = \mathbf{Z}$, and $\widetilde{H}_6(K(\mathbf{Z},4); \mathbf{Z}) = \mathbf{Z}_2$. Similarly the groups $\widetilde{\Omega}^{spin}_q(K(\mathbf{Z},4)))$ have been computed by Stong to be 0 for $0 \le q < 4$, and $\mathbf{Z}$ for $q = 4$. This is all we need for the present computation, which is moreover facilitated by considering the diagram in



Caption: The E^2 term in the spectral sequence computation of cobordism invariants relevant to the bilinear identity. Differentials act from the diagonal $p+q$ to the diagonal $p+q-1$. Since the cobordism groups are simple in low dimensions, the differentials are trivial and we can read off the answer.

The only nonzero group on the diagonal $p+q = 10$ is $E^2_{6,4} = 0$. All groups on the diagonals $p+q = 9, 11$ vanish. Thus we conclude

$$\Omega^{spin}_{10}(K(\mathbf{Z},4) \times K(\mathbf{Z},4), K(\mathbf{Z},4) \times \{*\} \cup \{*\} \times K(\mathbf{Z},4)) = \mathbf{Z}_2 \tag{3.20}$$

so there is only one nontrivial cobordism invariant!

One example of such an invariant is $\int_X a_1 \cup Sq^2 a_2$. In section 4.1 we give an example – $X = \mathbf{S}^2 \times \mathbf{S}^2 \times \mathbf{CP}^3$ – for which it is nonzero. So it is the unique invariant. $Q(a_1, a_2)$ is not identically zero, for we have seen that it coincides with $\int_X a_1 \cup Sq^2 a_2$ when a_1 and a_2 can be lifted to K-theory (which is the case for all classes on X). So $Q(a_1, a_2) = \int a_1 \cup Sq^2 a_2$ in general.

3.3. Parity Symmetry

We conclude this section by analyzing a symmetry of M-theory that has been obscured in our formalism. (One might return to this discussion after reading section 4.1.) Parity symmetry of M-theory acts by orientation reversal, together with $G \to -G$, which in terms of $a = G/2\pi + \lambda/2$ is $a \to -a + \lambda$. For $Y = \mathbf{S}^1 \times X$, there is hence a parity symmetry that acts by reflection of $\mathbf{S}^1$ together with $a \to -a + \lambda$. In Type IIA superstring theory, this symmetry is interpreted as $(-1)^{F_L}$. It plays an important role in the structure of the theory, and one does not expect it to be anomalous.

We will now demonstrate that there is an apparent anomaly, due to gravitational instantons, in the $(-1)^{F_L}$ symmetry in IIA theory. We will then show, as an application of the bilinear identity (3.13), that this anomaly is cancelled by a nontrivial effect in the RR sector.

Consider Type IIA superstring theory on a ten-dimensional spin manifold X. Denote by $q(V)$ the mod two index of the Dirac operator coupled to a real vector bundle V. (For greater precision, we will sometimes denote the mod 2 Dirac index on X with values in a real bundle V as $q(V; X)$.) In particular, the relevant mod two index for the Rarita-Schwinger operator is $q(TX)$. For a given set of background fields on X, let n_L and n_R denote the number of zero modes of the gravitino fields ψ_L and ψ_R coming from the left- or right-moving worldsheet Ramond sector; n_L and n_R are both congruent mod 2 to $q(TX)$. There is an effective action proportional to $\psi_L^{n_L} \psi_R^{n_R}$. As $(-1)^{F_L}$ acts as $\psi_L \to -\psi_L$, $\psi_R \to \psi_R$, the fermion measure μ transforms under $(-1)^{F_L}$ as

$$\mu \to (-1)^{q(TX)} \mu. \tag{3.21}$$

In the path integral the ghosts plus dilatino make no net contribution to the transformation of the measure (3.21) as they constitute an even number of chiral spin 1/2 fields. There is no problem in finding an X with $q(TX) \neq 0$; $X = \mathbf{HP}^2 \times T^2$ will serve as an example, as

we discuss more fully below. By deleting a point from X and projecting it to infinity, we get a candidate gravitational instanton in an asymptotically flat spacetime with a fermion measure that is odd under $(-1)^{F_L}$.

It appears that we must either search for a physical principle that might forbid this instanton, or accept that $(-1)^{F_L}$ is anomalous in asymptotically flat spacetime and hence in anything to which it can be compactified. Instead, we will demonstrate that this anomaly is canceled by an anomaly in the action of $(-1)^{F_L}$ on the Ramond-Ramond fields. We use the M-theory framework. The description by E_8 gauge theory does not make manifest how the partition function of the C-field transforms under a reflection of the circle together with $a \to \lambda - a$, but can be used to determine this transformation. The bilinear relation gives

$$f(\lambda) = f(a) + f(\lambda - a) + \int a \cup Sq^2(\lambda - a). \tag{3.22}$$

According to a result of Stong [7],

$$\int a \cup Sq^2 a = \int a \cup Sq^2 \lambda. \tag{3.23}$$

Hence,

$$(-1)^{f(\lambda - a)} = (-1)^{f(a)}(-1)^{f(\lambda)}. \tag{3.24}$$

So under parity or $(-1)^{F_L}$, the RR partition function is multiplied by $(-1)^{f(\lambda)}$. This factor will turn out to cancel the anomaly of the fermions. A quick way to demonstrate this cancellation is to use a relation to the heterotic string described at the end of this section. We will take a more leisurely route which brings out some useful information.

$f(\lambda)$ is defined as the mod 2 index of an E_8 bundle whose characteristic class is λ. Such a bundle can be described very simply: take the tangent bundle TX of X, whose structure group is $SO(10)$, and build from it an E_8 bundle using the chain of embeddings $SO(10) \subset SO(10) \times SU(4) \subset E_8$. The adjoint representation of E_8 decomposes under $SO(10) \times SU(4)$ as $(\mathbf{45}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{15}) \oplus (\mathbf{10}, \mathbf{6}) \oplus (\mathbf{16}, \mathbf{4})$. Since six copies of the $\mathbf{10}$ or four copies of the $\mathbf{16}$ will not contribute to the mod 2 index, and 15 copies of the $\mathbf{1}$ are equivalent to a single copy, we can replace the adjoint representation of E_8, for purposes of computing the mod 2 index, with the $\mathbf{45} \oplus \mathbf{1}$ of $SO(10)$. The bundle on X that corresponds to this representation is $\wedge^2 TX \oplus \mathcal{O}$, where $\mathcal{O}$ is a trivial line bundle and $\wedge^2 TX$ is the antisymmetric product of TX with itself. We thus have $f(\lambda) = q(\wedge^2 TX) + q(\mathcal{O})$, and hence, including

also the fermion anomaly from (3.21), the total sign change of the effective action under $(-1)^{F_L}$ is

$$J = (-1)^{q(\wedge^2 TX)+q(TX)+q(\mathcal{O})}. \tag{3.25}$$

We want to demonstrate that $J = 1$ for all ten-dimensional spin manifolds X. To do so, we will use the fact that J is a spin cobordism invariant; indeed, each factor in J is separately equal to 1 for any X that is the boundary of an eleven-dimensional spin manifold. The group Ω_{10}^{spin} is equal to $\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$. One choice of three independent invariants is $q(\mathcal{O})$, $q(TX)$, and

$$K = \int_X w_4 \cup w_6 = \int_X \lambda \cup Sq^2 \lambda. \tag{3.26}$$

(The examples we give will show that these three invariants are independent.)

One can pick two generators of the spin bordism group to be of the form $X = Y \times \mathbf{T}^2$, for suitable Y (with a supersymmetric or unbounding spin structure on $\mathbf{T}^2$). On such a manifold, $q(\mathcal{O}; X)$ is equal to the mod 2 reduction of $I(\mathcal{O}; Y)$. [7] This follows from the fact that a fermion zero mode on X is a constant on $\mathbf{T}^2$ times a zero mode on Y. Likewise, on such a manifold, $q(TX; X)$ equals the mod 2 reduction of $I(TY; Y)$, the ordinary index of the Dirac operator on Y with values in the tangent bundle. To show this, one uses the decomposition of the tangent bundle TX of X as $TX = TY \oplus \mathcal{O} \oplus \mathcal{O}$, where $\mathcal{O} \oplus \mathcal{O}$ is the tangent bundle to $\mathbf{T}^2$. The two copies of $\mathcal{O}$ do not contribute to the mod 2 index, so we need the mod two index of the Dirac operator on $Y \times \mathbf{T}^2$ with values in TY. As a zero mode must be a constant on $\mathbf{T}^2$ times a zero mode on Y, we get $q(TX; X) = I(TY; Y) \bmod 2$. Any manifold $Y \times \mathbf{T}^2$ has $K = 0$, since w_4 and w_6 are pullbacks from Y.

Now, let Y_1 be a spin manifold with Dirac index 1 and Rarita-Schwinger index 0, and let Y_2 be a spin manifold with Dirac index 0 and Rarita-Schwinger index 1. Such manifolds exist. The spin cobordism group in eight dimensions is known to be $\mathbf{Z} \oplus \mathbf{Z}$, generated by a manifold Y_1 such that $4Y_1$ is spin cobordant to $K3 \times K3$ and by $Y_2 = \mathbf{HP}^2$. Then $X_i = Y_i \times \mathbf{T}^2$, $i = 1, 2$, with unbounding (or RR) spin structure on $\mathbf{T}^2$, will serve as two generators of Ω_{10}^{spin}. For these generators, we have in view of the remarks in the last paragraph $q(\mathcal{O}; X_1) = 1$, $q(TX_1; X_1) = 0$, and $q(\mathcal{O}; X_2) = 0$, $q(TX_2; X_2) = 1$.

[7] We use $\mathcal{O}$ for a trivial real or complex line bundle. Which is meant should be clear from the context. In general, $I(x; X)$ denotes the ordinary index of the Dirac operator on X coupled to the K-theory class x.

For examples of this kind, we can readily use index theory to show that the total anomaly factor J is trivial. Indeed, by reasoning as above, we find that the mod 2 index on $X = Y \times \mathbf{T}^2$ with values in $\wedge^2 TX$ is the same as the ordinary index of the Dirac operator on Y with values in $\wedge^2 TY \oplus \mathcal{O}$. (We use the decomposition $\wedge^2 TX = \wedge^2 TY \oplus 2\,TY \oplus \mathcal{O}$, and note that two copies of TY do not contribute to the mod 2 index.) For such manifolds, the total anomaly factor becomes

$$J = (-1)^{I(\wedge^2 TY) + I(TY)}. \tag{3.27}$$

The index theorem shows that on any eight-dimensional spin manifold Y,

$$\begin{aligned} I(\mathcal{O}; Y) &= \int_Y \hat{A}_8 \\ I(TY; Y) &= \int_Y \left(248 \hat{A}_8 - \lambda^2\right) \\ I(\wedge^2 TY; Y) &= \int_Y \left(28 \hat{A}_8 + \lambda^2\right) \end{aligned} \tag{3.28}$$

In particular, $I(TY;Y)$ is congruent to $I(\wedge^2 TY; Y)$ mod 2. So for this class of manifold, there is no anomaly in $(-1)^{F_L}$.

For the third generator of Ω_{10}^{spin}, we can take a manifold $V_{1,1}$ defined as a hypersurface of degree $(1,1)$ in $\mathbf{CP}^2 \times \mathbf{CP}^4$. This manifold has $K = 1$. It also has, as we demonstrate in appendix B, $q(\mathcal{O}) = q(TX) = q(\wedge^2 TX) = 0$. So in particular, the total anomaly vanishes for this manifold. This completes the demonstration that the combined $(-1)^{F_L}$ anomaly of the fermions plus the RR fields always vanishes in Type IIA superstring theory at $G_2 = 0$. We extend the analysis to backgrounds with nonvanishing G_2 in section eight.

Let us now make more explicit what is going on in the above discussion for $X_2 = \mathbf{HP}^2 \times \mathbf{T}^2$, where an anomaly of the RR fields cancels a fermion anomaly. The first key point is that $H_4(X_2; \mathbf{Z}) = \mathbf{Z}$, generated by $U = \mathbf{HP}^1 \subset \mathbf{HP}^2$. Moreover, $\int_U \lambda = 1$. Hence, it is impossible for the four-form flux $\int_U G/2\pi$ to vanish. The quantization condition (as derived in [4] from world-volume anomalies) is

$$\int_U \frac{G}{2\pi} = \frac{1}{2} \int_U \lambda \bmod \mathbf{Z}. \tag{3.29}$$

So the least action G-field on X_2 has $\int_U G/2\pi = \pm 1/2$. The two possibilities are exchanged by parity or $(-1)^{F_L}$, and as they contribute to the partition function with a relative sign, there is an anomaly. If instead the λ class is divisible by two, then the G-field can vanish.

A configuration with $G = 0$, if it exists, is parity-invariant, and makes a nonvanishing contribution to the sum over G-field fluxes, so if such a configuration exists, there is no parity anomaly in the partition function of the M-theory C-field, or of the RR fields in the Type IIA description.

Application To The Heterotic String

The result we have just found has an interesting application to the $E_8 \times E_8$ heterotic string.

Consider the $E_8 \times E_8$ heterotic string on a ten-dimensional spin manifold X. Let a and b be the characteristic classes of the two E_8 bundles. Anomaly cancellation of the heterotic string requires that $a + b = \lambda$.

We want to show that the total number of fermion zero modes of the heterotic string is even. Otherwise, the heterotic string on X would be anomalous. Since the gluinos are chiral fermions in the adjoint representation of E_8, the number of gluino zero modes is $f(a) + f(b)$ mod 2. As we have proved above using Stong's formula, $f(a) + f(b) = f(a) + f(\lambda - a) = f(\lambda)$. The number of Rarita-Schwinger zero modes (including ghosts and dilatinos) is $q(TX)$ mod 2. So cancellation of the anomaly requires that $f(\lambda) = q(TX)$ mod 2, as we have shown above.

In fact, this result is a special case of a general statement about the heterotic string. In general, given any diffeomorphism $\varphi : X \to X$ with a lift of φ to the spin bundle of X and to the E_8 bundles, the effective action of the heterotic string is invariant under φ [14]. In case $\varphi = 1$ and we take φ to act on the spin bundle as multiplication by -1 and trivially on the E_8 bundles, invariance of the effective action is equivalent to the statement that the total number of fermion zero modes is even.

4. Topological Background

The present section is devoted to explaining topological background that will be important in the rest of the paper. Essentially everything in this section will be used later, though it is possible to proceed to section 5 without digesting everything presented here.

4.1. A Crash Course on Steenrod Squares

We have encountered a new topological operation – Sq^2 – that has not previously played much role in physics. We will here, with no attempt at completeness, try to motivate the properties of Steenrod squares that are needed in the present paper. Some references with further useful material and details are [18,19].

Suppose that X is a manifold and Q a codimension k submanifold whose normal bundle N is oriented. Let $b = [Q]$ be the Poincaré dual cohomology class to Q. b is a k-dimensional cohomology class and is represented in de Rham cohomology by a k-form delta function supported on Q and integrating to 1 over the normal directions to Q.

Let $w_i(N) \in H^i(Q; \mathbf{Z}_2)$ be the Stieffel-Whitney classes of N. We want to consider the objects that we can loosely call $w_i(N) \cup b \in H^{k+i}(X; \mathbf{Z}_2)$. Informally, this cup product makes sense because, while $w_i(N)$ is only defined in a small neighborhood of Q,[8] b anyway has its support in that neighborhood. The precise formal way to define $w_i(N) \cup b$ is as $i_*(w_i(N))$, where $i : Q \to X$ is the inclusion and i_* the push-forward. We will settle for informal expressions like $w_i(N) \cup b$.

Suppose we are given $b \in H^k(X; \mathbf{Z})$. We represent it as the Poincaré dual to a submanifold Q.[9] Then we define

$$Sq^i(b) = w_i(N) \cup b \in H^{k+i}(X; \mathbf{Z}_2). \tag{4.1}$$

It can be shown that, as an element of $H^{k+i}(X; \mathbf{Z}_2)$, $Sq^i(b)$ is independent of the choice of a submanifold Q dual to b. Not proving this will be a major gap in our presentation.

Thus, Sq^i, as we have defined it so far, is a linear map

$$Sq^i : H^k(X; \mathbf{Z}) \to H^{k+i}(X; \mathbf{Z}_2). \tag{4.2}$$

Sq^i is read "square i."

Actually, the Sq^i can be extended to linear maps $H^k(X; \mathbf{Z}_2) \to H^{k+i}(X; \mathbf{Z}_2)$. Given $\overline{b} \in H^k(X; \mathbf{Z}_2)$, one represents $\overline{b}$ as the Poincaré dual of a submanifold Q (whose normal

[8] It is defined to begin with on Q and can then be pulled back to a small tubular neighborhood of Q.

[9] In general, there are obstructions to doing this, but some odd multiple of b can always be so represented, and that is good enough for our purposes, since the Steenrod squares annihilate classes that are divisible by 2.

bundle is not necessarily orientable, so that its Poincaré dual is only defined mod 2) and defines again

$$Sq^i(\overline{b}) = w_i(N) \cup \overline{b}. \tag{4.3}$$

If $\overline{b}$ is the mod 2 reduction of an integral class b, then these definitions imply $Sq^i(b) = Sq^i(\overline{b})$.

The Sq^i for odd i can actually be defined as maps to $H^{k+i}(X;\mathbf{Z})$; $Sq^i(y)$ is always two-torsion, that is, $2Sq^i(y) = 0$. We will define the integer-valued Sq^i only for $i = 1$ (which we consider later) and $i = 3$. The Stieffel-Whitney class w_3 has a canonical integral lift W_3, which is the obstruction to Spin^c structure. It arises by considering the coefficient sequence

$$0 \to \mathbf{Z} \xrightarrow{2} \mathbf{Z} \xrightarrow{r} \mathbf{Z}_2 \to 0, \tag{4.4}$$

where the first map is multiplication by 2 and r is reduction mod 2. This leads to a long exact sequence

$$\dots H^2(Q;\mathbf{Z}) \xrightarrow{r} H^2(Q;\mathbf{Z}_2) \xrightarrow{\beta} H^3(Q;\mathbf{Z}) \xrightarrow{2} H^3(Q;\mathbf{Z}) \dots \tag{4.5}$$

where β is called the "connecting homomorphism" or the Bockstein. One defines

$$W_3(N) = \beta(w_2(N)). \tag{4.6}$$

Exactness of (4.5) implies that $2W_3(N) = 0$, and that $W_3(N) = 0$ precisely if $w_2(N)$ can be lifted to a class in $H^2(Q;\mathbf{Z})$. (This can be used to show that $W_3(N)$ is the obstruction to having a Spin^c structure on N.) To define

$$Sq^3 : H^k(X;\mathbf{Z}) \to H^{k+3}(X;\mathbf{Z}), \tag{4.7}$$

we simply use W_3 instead of w_3:

$$Sq^3(y) = W_3 \cup y. \tag{4.8}$$

It can be shown that W_3 reduces mod 2 to the Stieffel-Whitney class w_3, so Sq^3 understood as a map to $H^{k+3}(X;\mathbf{Z})$ reduces mod 2 to Sq^3 as defined previously. Because of relations such as this one, the different Sq^i maps are compatible, and it usually causes no confusion to use the same name Sq^i for slightly different maps.

The Sq^i have many useful properties that we will need. For example, suppose that b is a k-dimensional cohomology class, dual to a codimension k manifold Q. The normal bundle N has rank k, so $w_i(N) = 0$ for $i > k$. It follows that

$$Sq^i(b) = 0 \text{ for } i > k. \tag{4.9}$$

To derive the next property of Sq^i recall (see, e.g. [20]) that the Stieffel-Whitney classes of a bundle can be defined using obstruction theory. In this approach one finds that for N of real rank k, $W_k(N)$ is the Euler class for k odd, while $w_k(N)$ is the mod 2 reduction of the Euler class for k even. The Euler class of the normal bundle of Q is represented by the zero set of a generic section s of the normal bundle, or equivalently by $Q \cap Q'$ where Q' is obtained from Q by displacing it by the section s. Q' is homologous to Q, and $Q \cap Q'$ is dual to $b \cup b$. So

$$Sq^k(b) = b \cup b \tag{4.10}$$

for a k-dimensional class b. The cup product $b \cup b$ is also written as b^2, and this formula is actually the reason that the Sq^i are called "squares."

Next, let us work out a formula for $Sq^i(b \cup b')$, where b and b' are classes of degree k and k'. We suppose that b is Poincaré dual to Q and b' is Poincaré dual to Q', and that Q and Q' intersect transversely in a codimension $k + k'$ manifold Q''. Then Q'' is dual to $b \cup b'$, and the normal bundles on Q'' are related by

$$N(Q'') = N(Q)|_{Q''} \oplus N(Q')|_{Q''}. \tag{4.11}$$

From (4.11), it follows that

$$w_i(N(Q'')) = \sum_{j=0}^{i} w_j(N(Q)) \cup w_{i-j}(N(Q')). \tag{4.12}$$

Hence, we deduce the Cartan formula

$$Sq^i(y \cup y') = \sum_{j=0}^{i} Sq^j(y) \cup Sq^{i-j}(y'). \tag{4.13}$$

Next, suppose that X is an orientable manifold of dimension n. Given $\overline{b} \in H^{n-1}(X; \mathbf{Z}_2)$, we wish to show that

$$Sq^1(b) = 0. \tag{4.14}$$

In fact, $\overline{b}$ is dual to a compact one-manifold Q, which is inevitably a union of circles and hence orientable. The Stieffel-Whitney classes of X, restricted to Q, are related to those of Q and of the normal bundle N by

$$(1+w_1(X)+w_2(X)+\ldots) = (1+w_1(Q)+w_2(Q)+\ldots)(1+w_1(N)+w_2(N)+\ldots), \tag{4.15}$$

so for instance

$$w_1(X) = w_1(Q)+w_1(N), \quad w_2(X) = w_2(N)+w_1(Q)w_1(N)+w_2(Q). \tag{4.16}$$

Since X and Q are orientable, we have $w_1(X) = w_1(Q) = 0$; hence $w_1(N) = 0$, and (4.14) follows.

Now suppose that X is a spin manifold of dimension n and $\overline{b} \in H^{n-2}(X;\mathbf{Z}_2)$. We want to prove that

$$Sq^2(b) = 0. \tag{4.17}$$

b is dual to a not necessarily orientable two-manifold Q. Every two-manifold has $w_1(Q)^2 + w_2(Q) = 0$. Since X is spin ($w_1(X) = w_2(X) = 0$), the relations (4.16) imply $w_2(N) = 0$, and (4.17) follows. Next consider $b \in H^{n-3}(X;\mathbf{Z})$. b is dual to an oriented three-manifold Q; such a manifold has $w_1 = w_2 = w_3 = W_3 = 0$. So we get

$$Sq^1(b) = Sq^2(b) = Sq^3(b) = 0. \tag{4.18}$$

For a final result of this kind, consider $b \in H^{n-4}(X;\mathbf{Z})$. Then b is dual to a four-manifold, and as four-manifolds are Spin^c, one has

$$Sq^3(b) = 0. \tag{4.19}$$

Some of the above formulas have an important generalization known as the Wu formula [18]. In general, on any n-manifold X, for $\overline{b} \in H^{n-i}(X;\mathbf{Z}_2)$, one has $Sq^i(\overline{b}) = V_i \cup \overline{b}$ where V_i is a polynomial in Stieffel-Whitney classes known as the Wu class. From the above, we have $V_1 = w_1$ and $V_2 = w_1^2 + w_2$.

Suppose we are given $\overline{b} \in H^k(X;\mathbf{Z}_2)$, dual to a manifold Q. Then $w_1(N)$ vanishes if and only if N is orientable; but orientability of N is the condition that $\overline{b}$ is the reduction of an element of $H^k(X;\mathbf{Z})$. Indeed, Q is dual to an element b of $H^k(X;\mathbf{Z})$ (which can be reduced mod 2 to give an element $\overline{b}$ of $H^k(X;\mathbf{Z}_2)$) precisely if its normal bundle is

orientable. So $Sq^1(\bar{b}) = 0$ if $\bar{b}$ is the reduction mod 2 of an integral class. More generally, it can be shown that

$$Sq^1(\bar{b}) = \beta(\bar{b}), \tag{4.20}$$

where β is the Bockstein map $\beta : H^k(X;\mathbf{Z}_2) \to H^{k+1}(X;\mathbf{Z})$ derived from the coefficient sequence (4.4). It fits in the exact sequence

$$\ldots H^k(X;\mathbf{Z}) \xrightarrow{r} H^k(X;\mathbf{Z}_2) \xrightarrow{\beta} H^{k+1}(X;\mathbf{Z}) \ldots, \tag{4.21}$$

with r the mod 2 reduction. The relation $Sq^1 = \beta$ defines Sq^1 as a map

$$Sq^1 : H^k(X;\mathbf{Z}_2) \to H^{k+1}(X;\mathbf{Z}). \tag{4.22}$$

Exactness of the sequence (4.21) implies that for $\bar{b} \in H^k(X;\mathbf{Z}_2)$,

$$Sq^1(\bar{b}) = 0 \text{ if and only if } \bar{b} = r(b) \tag{4.23}$$

for some $b \in H^k(X;\mathbf{Z})$.

Let us consider the object $Sq^1 Sq^2(b)$ for $b \in H^k(X;\mathbf{Z})$. Using the definition $Sq^2(b) = w_2(N) \cup b$, this is $Sq^1 Sq^2(b) = Sq^1(w_2(N) \cup b)$. Using the Cartan formula and (4.23), we have $Sq^1(w_2(N) \cup b) = Sq^1(w_2(N)) \cup b = W_3(N) \cup b$ (where in the last step we use (4.6)). But $Sq^3(b) = W_3(N) \cup b$ and so we have for $b \in H^k(X;\mathbf{Z})$

$$Sq^3(b) = Sq^1 Sq^2(b). \tag{4.24}$$

Hence, $Sq^3(b) = 0$ if and only if there exists $c \in H^{k+2}(X;\mathbf{Z})$ with $c = Sq^2(b)$ mod 2. For (as we learned in the previous paragraph) this is the condition for Sq^1 to annihilate $Sq^2(b)$.

The relation $Sq^3 = Sq^1 Sq^2$ is a special case of a system of relations among products of Steenrod squares called the Adem relations. Another special case of the Adem relations that we will need is

$$Sq^3 Sq^3 = 0. \tag{4.25}$$

In fact, $Sq^3(Sq^3 y)$ is represented by $Sq^3(W_3 \cup y)$. Using the Cartan formula and the fact that Sq^1 annihilates an integral class, this is $Sq^3 W_3 \cup y + W_3 \cup Sq^3 y$, and vanishes since $Sq^3 W_3 = W_3 \cup W_3$, $Sq^3 y = W_3 \cup y$, and $2 Sq^3 W_3 = 0$.

We now have the tools to verify some assertions made in section 3.1. For X a spin manifold of dimension n, given $a \in H^k(X;\mathbf{Z})$, $a' \in H^{n-k-2}(X;\mathbf{Z})$, we want to show that

$$\int Sq^2 a \cup a' = \int a \cup Sq^2 a'. \tag{4.26}$$

For $k = 4$, $n = 10$, this was asserted in section 3.1. In fact, from (4.17) we have $Sq^2(a \cup a') = 0$. In view of the assertion in (4.23), $Sq^1 a = Sq^1 a' = 0$, so the Cartan formula (4.13) gives $Sq^2(a \cup a') = Sq^2 a \cup a' + a \cup Sq^2 a'$. Putting these facts together, we have $Sq^2 a \cup a' + a \cup Sq^2 a' = 0$ and (as Sq^2 is only defined mod 2) this implies (4.26).

Now let F be a complex vector bundle. We wish to show that

$$c_3(F) = c_1(F)c_2(F) + Sq^2 c_2(F) \text{ mod } 2. \tag{4.27}$$

For $c_1(F) = 0$, this assertion was made in section 2.2. In proving such a statement, it suffices, by the splitting principle,[10] to consider the case that $F = \oplus_{i=1}^n \mathcal{L}_i$ is a direct sum of line bundles. (This is proved by finding a fiber bundle Z over X, such that F pulls back on Z to a sum of line bundles, and such that if a cohomological statement like (4.27) holds on Z, it must also hold on X.) Let $b_i = c_1(\mathcal{L}_i)$. We have

$$\begin{aligned} c_1(F) &= \sum_i b_i \\ c_2(F) &= \sum_{i<j} b_i b_j \\ c_3(F) &= \sum_{i<j<k} b_i b_j b_k \\ Sq^2 c_2(F) &= \sum_{i \neq j} b_i^2 b_j \text{ mod } 2. \end{aligned} \tag{4.28}$$

The last of these formulas is proved using the Cartan formula together with $Sq^2(b_i) = b_i^2$. (4.27) is a straightforward consequence of these formulas.

Examples

Since our discussion has been somewhat abstract, we will here give a few examples.

First we want to give an example in which the bilinear form $\int a \cup Sq^2 b$ that appears in the bilinear relation for the E_8 mod 2 index is nontrivial. For this, quite elementary examples suffice. We take, for example, $X = \mathbf{S}^2 \times \mathbf{S}^2 \times \mathbf{CP}^3$. The second cohomology groups of the three factors are generated by classes d_1, d_2, d_3. The integral cohomology of X is generated by the d_i with the relations $d_1^2 = d_2^2 = 0$, $d_3^4 = 0$. Let $a = d_1 \cup d_3$, $b = d_2 \cup d_3$. We have $Sq^2(a) = d_1 \cup d_3^2$, $Sq^2(b) = d_2 \cup d_3^2$. (The right hand sides of these formulas are reduced mod 2 as Sq^2 is a map to the $\mathbf{Z}_2$-valued cohomology.) To prove these

[10] See, for example [21].

relations, one uses the Cartan formula, the fact that Sq^1 annihilates an integral class, and the fact that for d a two-dimensional class, $Sq^2(d) = d \cup d = d^2$. So we have

$$\int a \cup Sq^2 b = \int Sq^2 a \cup b = \int d_1 \cup d_2 \cup d_3^3 = 1 \tag{4.29}$$

Note that $Sq^3(a) = Sq^3(b) = 0$, as $Sq^2(a)$ and $Sq^2(b)$ have integral lifts, namely $d_1 \cup d_3^2$ and $d_2 \cup d_3^2$. In this example, all of the cohomology of X can be lifted to K-theory, since the cohomology ring is generated by two-dimensional classes, and any two-dimensional class can be lifted to K-theory by finding a suitable line bundle.

It is a bit trickier to give an example in which Sq^3 is nontrivial. Take $X = \mathbf{RP}^7 \times \mathbf{RP}^3$. Then the $\mathbf{Z}_2$-valued cohomology of the two factors are generated, respectively, by one-dimensional classes a and b with $a^8 = b^4 = 0$. Let $h = Sq^1(a \cup b)$, where we understand Sq^1 as a map to the integral cohomology of X, so $h \in H^3(X; \mathbf{Z})$. We can evaluate the mod 2 reduction of h by interpreting Sq^1 as a map to the $\mathbf{Z}_2$-valued cohomology. With this interpretation of Sq^1, we have $Sq^1(c) = c \cup c = c^2$ for any one-dimensional class c. Using this and the Cartan formula, we see that the mod 2 reduction of h is $a^2 \cup b + a \cup b^2$, so in particular h is nonzero. We can also evaluate the mod 2 reduction of $h \cup h$; it is $a^4 \cup b^2 \neq 0$, so in particular $h \cup h$ is nonzero. [11] As $h \cup h = Sq^3 h$, we see that $Sq^3 h \neq 0$.

We want to give an example of Sq^3 acting nontrivially on $H^4(X; \mathbf{Z})$. For this, we begin with a five-manifold Q constructed as a $\mathbf{CP}^2$ bundle over $\mathbf{S}^1$, with $\mathbf{CP}^2$ undergoing complex conjugation as one goes around the $\mathbf{S}^1$. This example was discussed in [22], and is not Spin^c, that is $W_3(Q) \neq 0$. (In fact, in this example, $W_3(Q)$ is Poincaré dual to $L = \mathbf{CP}^1 \times p$, where p is a point in $\mathbf{S}^1$ and $\mathbf{CP}^1$ a linearly embedded subspace of $\mathbf{CP}^2$.) One can construct an $SO(3)$ bundle N over Q such that the total space U of the bundle is spin. (Explicitly, one can take N to be the direct sum of the nontrivial real line bundle over $\mathbf{S}^1$ and the standard complex line bundle $\mathcal{O}(1)$ over $\mathbf{CP}^2$ regarded as a rank two real bundle.) Now, embed Q in U as the zero section and let $h \in H^3(U; \mathbf{Z})$ be Poincaré dual to Q. The normal bundle to Q is N, and as $W_3(N) = W_3(Q) \neq 0$, one has $Sq^3(h) \neq 0$; in fact, $Sq^3(h)$ is Poincaré dual to L (embedded in U via $L \subset Q \subset U$; note that as a submanifold of U, L has codimension six and so is dual to a degree six cohomology class) because in the cohomology of Q, L is dual to $W_3(N)$.

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[11] However, $h \cup h$ is two-torsion. Indeed, given any two integral classes h and h' of odd degree, one has $h \cup h' = -h' \cup h$ or $h \cup h' + h' \cup h = 0$. Setting $h' = h$, we get $2h \cup h = 0$, so $h \cup h$ is always two-torsion.

Now set $X = U \times \mathbf{T}^2$ and $a = h \cup b$, where $b \in H^1(\mathbf{T}^2;\mathbf{Z})$ is any class not divisible by two. So $a \in H^4(X;\mathbf{Z})$. As all Sq^i annihilate b (Sq^1 annihilates b as b is an integral class, and the higher Sq^i do so for dimensional reasons), we have by the Cartan formula $Sq^3 a = h \cup h \cup b \neq 0$. In this example, X is not compact. If desired, one can compactify X without modifying the discussion by adding a point at infinity to each $\mathbf{R}^3$ fiber of $U \to Q$, replacing the $\mathbf{R}^3$ bundle by an $\mathbf{S}^3$ bundle.

Finally, we mention that if one does not wish to restrict to ten-manifolds, there is a set of "universal" examples, namely the cohomology of the Eilenberg-MacLane spaces $K(\mathbf{Z}, n)$ themselves. They are "universal" because any cohomology class on X is uniquely associated to the homotopy class of a map $f : X \to K(\mathbf{Z}, n)$. The cohomology of the spaces $K(\mathbf{Z}, n)$ is built from some basic generators and certain "cohomology operations" such as Sq^i.

4.2. Torsion Pairings

We will here describe another important bit of topological background.

We work on an oriented manifold X of dimension n. For $a \in H^k(X;\mathbf{Z})$, $c \in H^{n-k}(X;U(1))$, there is a cup product $a \cup c \in H^n(X;U(1)) = U(1)$. This gives a pairing which we denote as

$$(a, c) = \int_X a \cup c. \tag{4.30}$$

One version of Poincaré duality is the statement that this pairing is a Pontryagin duality between $H^k(X;\mathbf{Z})$ and $H^{n-k}(X;U(1))$.

Now consider the short exact sequence of coefficient groups

$$0 \to \mathbf{Z} \xrightarrow{i} \mathbf{R} \xrightarrow{r} U(1) \to 0. \tag{4.31}$$

Here i is the embedding of $\mathbf{Z}$ in $\mathbf{R}$, and r maps the real number t to $\exp(2\pi i t) \in U(1)$. The associated cohomology sequence reads

$$\cdots \to \ H^s(X;\mathbf{R}) \xrightarrow{r} H^s(X;U(1)) \xrightarrow{\beta} H^{s+1}(X;\mathbf{Z}) \xrightarrow{i} H^{s+1}(X;\mathbf{R}) \ \to \cdots. \tag{4.32}$$

A class $b \in H^{s+1}(X;\mathbf{Z})$ is torsion if and only if $i(b) = 0$. Exactness of (4.32) says that this is the condition for the existence of $c \in H^s(X;U(1))$ with $\beta(c) = b$.

Now suppose we are given $a \in H^k(X;\mathbf{Z})$ and a torsion class $b \in H^{n-k+1}(X;\mathbf{Z})$. Because b is torsion, there exists $c \in H^{n-k}(X;U(1))$ such that $\beta(c) = b$. In general, the pairing $\int_X a \cup c$ depends on the choice of c and not only on b. However, the indeterminacy in

c is (according to (4.32)) $c \to c + r(e)$ where $e \in H^{n-k}(X; \mathbf{R})$. Because the cup product of a torsion class with a real class is zero, the pairing (a, c) is unaffected by the indeterminacy in c if a is a torsion class.

So for torsion classes a and b, there is a well-defined torsion pairing

$$T : H^k_{tors}(X; \mathbf{Z}) \times H^{n-k+1}_{tors}(X; \mathbf{Z}) \to U(1), \tag{4.33}$$

defined by $T(a, b) = \int a \cup c$ where $\beta(c) = b$. Equivalently, a being torsion, there is c' with $\beta(c') = a$, and we can define $T(a, b) = \int c' \cup b$. Poincaré duality can be used to prove that T is a Pontryagin duality between H^k_{tors} and H^{n-k+1}_{tors}. In the text below we often switch between "additive" and "multiplicative" notation for abelian groups. When we use additive notation we will consider T to be valued in $\mathbf{R}/2\pi\mathbf{Z}$. Which convention is used will be clear from the context.

Here is a typical example where we will use the torsion pairings. For X of dimension 10 and $a, b \in H^4(X; \mathbf{Z})$, we have met in section 3 the bilinear form

$$\phi(a, b) = \int_X a \cup Sq^2 b. \tag{4.34}$$

If a is a torsion class, then ϕ can be interpreted as a torsion pairing, as follows. We have $\beta(Sq^2 b) = Sq^1 Sq^2 b = Sq^3 b$ by the Adem relations. Now, $Sq^3 b$ is a torsion class, and running through the definition of T, we see that for a torsion, we have $\phi(a, b) = T(a, Sq^3 b)$. We have already proved that $\phi(a, b)$ is symmetric, so it follows that T is symmetric; for torsion classes a, b, we have

$$T(a, Sq^3 b) = T(b, Sq^3 a). \tag{4.35}$$

We conclude with some technical observations that will be useful in sections 6 and 7. For a a torsion class in $H^4(X; \mathbf{Z})$ and b any class in $H^4(X; \mathbf{Z})$, consider $< a, b >= T(a, Sq^3 b)$. We will show that $< a, b >$ establishes a duality between certain spaces. Note that though T is $U(1)$-valued in general, as $Sq^3 b$ is two-torsion, $T(a, Sq^3 b)$ takes values in the subgroup $\{\pm 1\}$ of $U(1)$, which is isomorphic to $\mathbf{Z}_2$; so we will consider $T(a, Sq^3 b)$ to be $\mathbf{Z}_2$-valued.

Let $A = Sq^3(H^4(X; \mathbf{Z}))$, that is, A is the subgroup of $H^7(X; \mathbf{Z})$ consisting of elements of the form $Sq^3 b$ for $b \in H^4(X; \mathbf{Z})$. Let $B = Sq^3(H^4_{tors}(X; \mathbf{Z}))$; that is, B is the subgroup of A consisting of elements $Sq^3 b$ where b is torsion.

Let Υ_0 be the kernel of $Sq^3 : H^4_{tors}(X; \mathbf{Z}) \to H^7(X; \mathbf{Z})$. It consists of torsion classes b such that $Sq^3 b = 0$, i.e., such that $Sq^2 b$ has an integral lift. Υ_0 has a subspace that we

will call Υ, which consists of torsion classes b such that $Sq^2 b$ has an integral lift which moreover is *torsion*. As Υ is a subspace of Υ_0, $V = H^4_{tors}/\Upsilon$ has $W = H^4_{tors}/\Upsilon_0$ as a quotient: $W = V/(\Upsilon_0/\Upsilon)$.

A, B, V, and W are all vector spaces over the field $\mathbf{Z}_2$. (For A and B this is obvious; for V and W it requires the observation that for any torsion class c, $2c \in \Upsilon$, since $Sq^2(2c) = 0$.) Moreover, since Sq^3 (by definition) maps W injectively to $H^7(X;\mathbf{Z})$, W is isomorphic to its image, which is B.

The pairing $< a, b >$ is nondegenerate as a map $W \times W \to \mathbf{Z}_2$ or equivalently $W \times B \to \mathbf{Z}_2$. For this, we just need to know that for every torsion class a with $Sq^3 a \neq 0$ (so that a represents a nonzero element of W), there is a torsion class b with $< a, b > \neq 0$. Since $< a, b >= T(Sq^3 a, b)$ for a and b torsion, this is true by nondegeneracy of the torsion pairings. Thus, there is a natural duality (as well as a natural isomorphism) between B and W.

We claim that in addition, V is dual to A by the pairing that to $a \in V$ and $Sq^3 c \in A$ assigns the value $T(a, Sq^3 c)$. (This pairing is well-defined because if $a \in \Upsilon$, so $Sq^2 a$ can be lifted to a torsion integral class, then for any integral class c, $0 = \int Sq^2 a \cup c = \int a \cup Sq^2 c = T(a, Sq^3 c)$.) First, we must show that for all $a \in V$, there is $Sq^3 c \in A$ with $T(a, Sq^3 c) \neq 0$. If $Sq^3 a \neq 0$, the nondegeneracy of the torsion pairing gives us a torsion class c with $0 \neq T(Sq^3 a, c) = T(a, Sq^3 c)$. If $Sq^3 a = 0$ and a is a nonzero element of V, then $Sq^2 a$ can be lifted to an integral class, but this class cannot be chosen to be a torsion class (or to be congruent mod 2 to a torsion class). So by ordinary integer-valued Poincaré duality, there is $c \in H^4(X;\mathbf{Z})$ (not a torsion class) with $\int Sq^2 a \cup c \neq 0$ mod 2, and hence $\int a \cup Sq^2 c \neq 0$. This last expression equals $T(a, Sq^3 c)$. This shows that for any nonzero $a \in V$, there is $Sq^3 c \in A$ with $T(a, Sq^3 c) \neq 0$. Conversely, given any $c \in H^4(X;\mathbf{Z})$, if $Sq^3 c \neq 0$, then there exists $a \in H^4_{tors}(X;\mathbf{Z})$ such that that $T(a, Sq^3 c) \neq 0$.

So to summarize, V is dual to $A = Sq^3(H^4(X;\mathbf{Z}))$, and the quotient space $W = V/(\Upsilon_0/\Upsilon)$ of V is dual to the subspace $B = Sq^3(H^4_{tors}(X;\mathbf{Z}))$ of A. This induces a duality

$$(\Upsilon_0/\Upsilon)^* \cong A/B. \tag{4.36}$$

According to the definitions, Υ_0/Υ consists of torsion classes c that obey $Sq^3 c = 0$ and so can be lifted to K-theory, modulo those whose lift to K-theory is a torsion class. In other words, given $c \in \Upsilon_0$, we have $c \in \Upsilon$ if and only if $Sq^2 c$ can be lifted to an integral torsion class d (which can be the third Chern class of a K-theory lift of c).

4.3. A Note on Poincaré Duality In K-Theory

The result (4.36) has a relation to Poincaré duality (not used in the rest of the paper) that we will briefly explain.

In cohomology theory, one has the lattices $S = H^4(X;\mathbf{Z})/H^4_{tors}$ and $T = H^6(X;\mathbf{Z})/H^6_{tors}$. They are dual to each other by Poincaré duality. In passing to K-theory, one loses certain classes in S that are not annihilated by Sq^3. S is replaced by a sublattice S' (defined presently) that is of some finite index n. Poincaré duality holds in K-theory just as it does in cohomology. To maintain the duality between S and T, if one loses classes in S, one must gain classes in T; T must be replaced by a lattice T', containing T, such that T'/T has the same index as S/S'. In fact, T'/T must be dual to S/S'.

How does this happen? We will give a brief explanation, without any attempt at completeness. The analog of S in K-theory is the group S' of classes in $K(X)$ that are torsion if restricted to the three-skeleton of X modulo classes that are torsion if restricted to the four-skeleton. Given $a \in S$, a corresponds to an element of S' if and only if, after possibly adding to a a suitable torsion class, one can achieve $Sq^3 a = 0$ so that a can be lifted to K-theory. The class in S' determined by a is invariant under adding a torsion class to a (and vanishes if a is torsion). Thus $S/S' = A/B$.

The analog of T in K-theory is the group T' of classes that are torsion if restricted to the five-skeleton of X mod classes that are torsion if restricted to the six-skeleton. If $a \in H^4(X;\mathbf{Z})$ is torsion and can be lifted to K-theory, but one cannot take its lift to be torsion (thus, $Sq^2 a$ can be lifted to an integral class, but not a torsion integral class), then the lift of a to K-theory corresponds to an element of T', though a does not correspond to an element of T. T is the sublattice of T' consisting of elements that are trivial (and not just torsion) if restricted to the five-skeleton.

As a result, one has $T'/T = \Upsilon_0/\Upsilon$. Given the conventional Poincaré duality between S and T and the K-theory duality between S' and T', the duality between Υ_0/Υ and A/B is a consequence.

We have described a mechanism for "losing" classes in S in going to K-theory, and for "gaining" classes in T. There is no analogous gain of classes in S in going to K-theory, because every torsion element of $H^2(X;\mathbf{Z})$ can be lifted to torsion in K-theory (by finding a suitable line bundle). There is no analogous loss of classes in T because of (4.19). There is no further mechanism (involving higher AHSS differentials) for "losing" classes in S because, on dimensional grounds, there is no further mechanism for "gaining" classes in T.

5. Type II Superstrings, Cohomology, and K-Theory

We encountered Steenrod squares in analyzing M-theory phases in section 3, but they also enter in Type II superstring theory. Understanding this will help us understand what we should aim for in analyzing the M-theory partition function in section 6.

5.1. Role of Sq^3

We first think in terms of D-branes. Given $b \in H^k(X;\mathbf{Z})$, we want to ask if there exists a D-brane state such that its RR k-form charge is b and the r-form charges vanish for $r < k$. For this, we pick a submanifold Q of spacetime that is Poincaré dual to b, and try to wrap a D-brane on Q. Such a D-brane state will automatically have the desired k-form charge, and, as Q is of codimension k, it will have vanishing r-form charges for $r < k$. Depending on the Chan-Paton gauge field on the brane, there may be RR s-form charges for $s > k$.

There is, however, an obstruction to wrapping a D-brane on Q [22]: such a D-brane exists if and only if the normal bundle to Q (or equivalently, as X is spin, Q itself) is Spin^c. In other words, the condition is $W_3(Q) = 0$. But $W_3(Q) = 0$ implies $Sq^3(b) = 0$. In other words, a D-brane whose lowest nonvanishing RR charge is b exists only if $Sq^3(b) = 0$.

Since D-brane charge is measured by K-theory, finding a D-brane whose lowest nontrivial brane charge is $b \in H^k(X;\mathbf{Z})$ means finding a K-theory class x (x is in $K(X)$ or $K^1(X)$ for even or odd k) such that x is trivial on the $(k-1)$-skeleton of X, and the obstruction to trivializing it on the k-skeleton is measured by b. We call such an x a lift of b to K-theory. The Atiyah-Hirzebruch spectral sequence (AHSS) gives a systematic framework for relating cohomology to K-theory and determining what cohomology classes can be lifted to K-theory [23]. In the AHSS, the first approximation to $K(X)$ is (for X of dimension n)

$$E_2 = \oplus_{2s \leq n} H^{2s}(X;\mathbf{Z}), \tag{5.1}$$

and the first approximation to $K^1(X)$ is

$$E_2^1 = \oplus_{2s+1 \leq n} H^{2s+1}(X;\mathbf{Z}). \tag{5.2}$$

Thus the starting approximation is the one in which D-brane charge is just measured by cohomology. Then one considers $Sq^3 : E_2 \leftrightarrow E_2^1$. Since $(Sq^3)^2 = 0$ (as we mentioned in (4.25)), one can define the cohomology groups of Sq^3 acting on E_2 and E_2^1, respectively. We call these cohomology groups E_3 and E_3^1; they give the second AHSS approximations

to $K(X)$ and $K^1(X)$, respectively. In the AHSS, there is a sequence of higher order corrections, converging eventually to a "graded version" of $K(X)$ and $K^1(X)$. (This is explained in detail in appendix C.) They are constructed from a series of "differentials" $d_r : H^k(X;\mathbf{Z}) \to H^{k+r}(X;\mathbf{Z})$, where $r = 3, 5, 7, \ldots$ and the first differential is $d_3 = Sq^3$. The image of d_r consists of torsion classes that in general have p-primary pieces for all primes that divide $(r+1)!$ For example, d_3 is annihilated by multiplication by 2 (as $2Sq^3 = 0$), and d_5 has 2-torsion and 3-torsion.

On dimensional grounds, the only higher AHSS differential that might be nontrivial on a ten-manifold is d_5. However, considerations of Poincaré duality show that d_5 annihilates the even-dimensional cohomology of a ten-manifold X (see the last sentence in section 4.3), so d_5 is not very important for understanding the Type IIA K-theory theta function studied in the present paper. It is possible that d_5 will have a nontrivial action on $H^3(X;\mathbf{Z})$, in which case it would play a role in understanding the K^1 theta function of Type IIB. Classes of the form $d_5(x)$ in general have both 2-torsion and 3-torsion. It can be shown that the 2-primary part d_5' is of order 4 and that $(2d_5')(x) = Sq^5(x)$ (d_5' itself is defined in terms of "secondary operations") while the 3-torsion part of $d_5(x)$ involves a mod-3 version of the Steenrod operations [24].

Differentials beyond d_5 vanish on a ten-manifold. Indeed, any class b in $H^1(X;\mathbf{Z})$ or $H^2(X;\mathbf{Z})$ can be lifted to K-theory. For instance, for $b \in H^2(X;\mathbf{Z})$, we can find a complex line bundle $\mathcal{L}$ with $c_1(\mathcal{L}) = b$, and then $\mathcal{L} - \mathcal{O}$ (with $\mathcal{O}$ a trivial line bundle) will do as a K-theory lift of b. So we only have to consider $b \in H^k(X;\mathbf{Z})$ for $k \geq 3$. Then $d_r b = 0$ for $r \geq 7$, since there is no torsion in $H^{10}(X;\mathbf{Z})$ and the higher cohomology of X is zero.

Since RR fields, like RR charges, are classified by K-theory, there is an analog of all this for RR fields. In fact, this analog will play the major role in the present paper. The following example will enable us to tie together some of the points that we have explained. Consider Type IIA superstring theory and ask whether, for some given $b \in H^4(X;\mathbf{Z})$, there exists an RR field with $G_0 = G_2 = 0$, and $G_4/2\pi = b$. For this, we must find a K-theory lift of b. Equivalently, we must find a complex vector bundle E with $c_1(E) = 0$, $c_2(E) = -b$; then the desired RR field is associated with the K-theory class $x = (E, F)$ (or $E - F$), where F is a trivial bundle with the same rank as E. For in this case $\sqrt{\hat{A}}\,\mathrm{ch}(x) = b + \ldots$ where the $\ldots$ are classes of degree ≥ 6 and we have mapped b into $H^4(X;\mathbf{Q})$ (thus losing torsion information). The role of subtracting F is to cancel G_0; G_2 is zero because $c_1(E) = 0$. If E exists, then $c_3(E)$ is an integral class with $c_3(E) = Sq^2 c_2(E) = Sq^2 b$ mod 2; we call

it an integral lift of $Sq^2 c_2(E)$. The existence of such an integral lift of $Sq^2 b$ means that $Sq^1 Sq^2 b = 0$ or

$$Sq^3 b = 0. \tag{5.3}$$

So we see again in this particular example that $Sq^3 b$ is an obstruction to lifting a cohomology class to K-theory.

5.2. Instability Of Some D-Branes

We will now give an application of this formalism to exhibit a new physical effect involving D-branes. We know [22] that certain D-branes that would be allowed if D-brane charge were measured by cohomology are actually not allowed because D-brane charge is really measured by K-theory. We will now show a flip side to this: certain D-branes that do exist and would be stable if D-brane charge were measured by cohomology are actually unstable, in fact, they are in the topologically trivial component of the field space.

Suppose we are given $c_0 \in H^{k-3}(X; \mathbf{Z})$ with $Sq^3 c_0 \neq 0$, and set $c = Sq^3 c_0$. To be definite in the terminology, we will assume that $k = 2n$ is even. Now the relation

$$Sq^3 c_0 = c \tag{5.4}$$

can be read in two ways. It asserts that c_0, not being annihilated by Sq^3, is not an element of the cohomology of Sq^3, and so cannot be lifted to an element of $K^1(X)$. But it also says that c, while annihilated by Sq^3 (since $Sq^3 Sq^3 = 0$), is trivial as an element of the Sq^3 cohomology – so c can be lifted to K-theory but the lift is zero. The first statement means that c_0 is not the lowest brane charge of any D-brane. In trying to construct such a D-brane, one would run into the anomaly studied in [11], which we will soon look at from a different point of view. The second statement means that while there exists a D-brane whose lowest brane charge is c, this D-brane is unstable. This last statement is the novel one that we now wish to explain.

First let us recall a standard construction of a K-theory class on a sphere $\mathbf{S}^{2n}$. It is equivalent to construct a K-theory class on $\mathbf{R}^{2n}$ with a trivialization at infinity. For this we take a pair (E, F) of trivial bundles of equal rank $N = 2^{n-1}$, together with the usual tachyon condensate (first considered in various examples in [25]):

$$T = \frac{\vec{\Gamma} \cdot \vec{x}}{\sqrt{1 + |\vec{x}|^2}}. \tag{5.5}$$

Near infinity, T is a unitary matrix that defines a generator of $\pi_{2n-1}(U(N))$; as a result, the pair (E,F) with this tachyon condensate at infinity is a generator of the compactly supported K-theory of $\mathbf{R}^{2n}$ or equivalently a generator of $K(\mathbf{S}^{2n})$.

Now let us recall how this works in a global situation. (The following construction is due to Atiyah, Bott, and Shapiro [26]; for an explanation for physicists, see section four of [27].) We start with a class $c \in H^{2n}(X;\mathbf{Z})$ and find a Spin^c manifold Q dual to c. We want to describe a D-brane wrapped on Q in terms of K-theory. For this, we lift c to K-theory by constructing a suitable K-theory class supported near Q. We set $E = S_+(N)$, $F = S_-(N)$ where $S_\pm(N)$ are positive and negative chirality Spin^c bundles of the normal bundle N of Q. We pull back E and F to a small neighborhood W of Q in X; topologically, we can think of W as the total space of the normal bundle N. Then in each fiber of $W \to Q$, we use the formula (5.5) to define the tachyon field. This describes the D-brane state near Q with a trivialization (tachyon condensation that brings us to the vacuum) away from the immediate neighborhood of Q. Denote by W' the neighborhood with Q omitted, and similarly, denote by X' the complement of Q in X. For a complete description, one extends E and F over X (perhaps after a replacement $(E,F) \to (E\oplus G, F\oplus G)$ for some bundle G) in such a way that T extends over X' as a unitary map $T : E \to F$. The importance of extending T is that if one cannot extend T over X', one will end up with additional D-branes somewhere else away from Q, where unitarity of T breaks down.

Using an additional bit of physics, the discussion we are about to give can be simplified somewhat. The simplifying fact is that actually, a D-brane system wrapped on Q with bundles (E,F) is only allowed if the bundles E and F are isomorphic when restricted to Q. Otherwise, one cannot solve the equations for the RR fields [2]. (This is a K-theory version of a statement at the level of cohomology that the Euler class to the normal bundle of a brane must vanish [28]; otherwise, the equation for the appropriate RR or NS p-form field that couples magnetically to the brane in question would have no solution.) This means that, after possibly replacing (E,F) by $(E\oplus G, F\oplus G)$, we can assume that E and F are trivial in W. As a result, we can interpret the tachyon field topologically as a map $T : W' \to U(N)$ (for some large N). Thus, the whole content of the D-brane state is captured by a $U(N)$-valued function on W', just as in the example on $\mathbf{R}^{2n}$, it was captured by a $U(N)$-valued function on the complement of the origin in $\mathbf{R}^{2n}$.

Now let us ask under what conditions a D-brane wrapped on Q, constructed as above, can decay even though the homology class of Q may be nontrivial. This can happen if the K-theory class of the D-brane is zero. If so, the D-brane can decay, by a process that

involves nucleation of $9-\overline{9}$-brane pairs in the intermediate state, to exploit the fact that, modulo creation and annihilation of such pairs, D-brane states are completely classified by their class in $K(X)$.

The D-brane wrapped on Q is trivial in $K(X)$ if the map $T : W' \to U(N)$ can be extended to a map $T : X' \to U(N)$. For in this case, we can extend E and F as trivial bundles over X' while also extending the tachyon field as a unitary map between them. We end up with a trivial class $(E,F) \in K(X)$ since E and F are both trivial. By contrast, if T did not extend over X' as a map to $U(N)$, then to extend T we would need to extend E and F over X' as suitable nontrivial bundles (chosen so that T can be extended), and we would end up with a nonzero K-theory class (E,F). That is what happens for stable D-brane states.

Extension Of T

Under what conditions can we extend $T : W' \to U(N)$ to $T : X' \to U(N)$? As we will see, this will happen if there is $c_0 \in H^{2n-3}(X;\mathbf{Z})$ with $Sq^3 c_0 = c$. In fact, as c_0 is an odd degree cohomology class, one can try to lift it to an element of $K^1(X)$. The lift will fail, as $Sq^3 c_0 \neq 0$, and the failure will give us, as we will see momentarily, the desired extension of T over X'.

An element of $K^1(X)$ can be described by a map $V : X \to U(N)$ (for some large N). Let us try to construct such a map associated with c_0. We will use obstruction theory (see [14] for a review for physicists). We begin by triangulating X. The class $c_0 \in H^{2n-3}(X;\mathbf{Z})$ defines (up to a certain equivalence relation) an integral-valued function on the set of $(2n-3)$-simplices; this function adds up to zero for any collection of $(2n-3)$-simplices that comprise the boundary of a $(2n-2)$-simplex.

We define V inductively on the p-skeleton (the union of all the p-simplices) for $p = 0, 1, 2, \ldots$. At the p-th step, V has been defined on the $(p-1)$-skeleton, and we wish to define it on the p-skeleton. Each p-simplex is topologically a p-dimensional ball B^p with boundary $\mathbf{S}^{p-1}$ made from $(p-1)$-simplices; V has already been defined on the boundary. If V, restricted to the boundary, is non-trivial in $\pi_{p-1}(U(N))$, it has no extension over B^p. If V is trivial on the boundary, its extensions over B^p are classified by $\pi_p(U(N))$.

To begin the induction, we define V to be identically 1 on the $(2n-4)$-skeleton. To extend V on the $(2n-3)$-skeleton, we need an element of $\pi_{2n-3}(U(N)) = \mathbf{Z}$ for each $(2n-3)$ simplex B. We simply assign to B the integer determined by the cohomology class c_0. In extending V over the $(2n-2)$-skeleton, there is potentially an obstruction

since $\pi_{2n-3}(U(N)) \neq 0$. However, the obstruction vanishes because c_0 assigns the value 0 to a sum of $(2n-3)$-simplices that make up the boundary of a $(2n-2)$-simplex. At the next step, there is no obstruction to extending V over the $(2n-1)$-skeleton, since $\pi_{2n-2}(U(N)) = 0$. In extending V over the $2n$-skeleton, however, there is a potential obstruction, associated with $\pi_{2n-1}(U(N)) = \mathbf{Z}$. The obstruction assigns an integer to each $2n$-simplex.

It can be shown that this collection of integers defines an element of $H^{2n}(X;\mathbf{Z})$. Moreover, this element is just $Sq^3 c_0$. This assertion is equivalent to the statement that $Sq^3 c_0$ is the first obstruction to lifting c_0 to an element of $K^1(X)$. We have denoted $Sq^3 c_0$ as c. The dual to c is our manifold Q, and having c as the obstruction to extending V means that V can be extended over the complement X' of Q.

Thus, V is the desired extension of T whose existence shows that a D-brane wrapped on Q can be unstable. We have not above defined the topological type of T in a completely unique way, because (using different Spin^c structures) there can be different D-brane states wrapped on Q. They differ in their $(2n+2k)$-form charges for $k \geq 1$. However, V coincides with *some* $U(N)$ valued function T whose behavior near Q is suitable to describe a D-brane wrapped on Q. Existence of V means that this D-brane state is unstable. Other D-branes wrapped on Q, if they carry $(2n+2k)$-form charges that cannot be written as $Sq^3(\ldots)$, are not completely unstable but can decay to D-branes wrapped on manifolds of dimension less than that of Q.

Examples

We will conclude by giving a few concrete examples of D-branes wrapped on non-trivial homology cycles that are nonetheless unstable. Pursuing one of the examples considered in section 4.1, we take $X = \mathbf{RP}^7 \times \mathbf{RP}^3$, with generators a and b for $H^1(\mathbf{RP}^7;\mathbf{Z}_2)$ and $H^1(\mathbf{RP}^3;\mathbf{Z}_2)$. $Sq^1 a$ and $Sq^1 b$ are two-torsion integral classes that we will somewhat loosely call a^2 and b^2. (Strictly speaking, as a and b are mod 2 classes, their squares $a^2 = a \cup a$ and $b^2 = b \cup b$ are mod 2 classes; these have integral lifts that we will also call a^2 and b^2.) We set $c = a^4 \cup b^2$. c is dual to $B = \mathbf{RP}^3 \times \mathbf{RP}^1$, with the two factors linearly embedded in the two factors of X. As B is nontrivial in homology, it appears that a D-brane wrapped on B would be stable. But actually, we have $c = h \cup h = Sq^3 h$, where $h = Sq^1(a \cup b)$ reduces mod 2 to $a^2 \cup b + a \cup b^2$. So some $D3$-brane wrapped on B is unstable.

Similarly, we could set $c' = a^6 \cup b^2$, which is dual to $B' = \mathbf{RP}^1 \times \mathbf{RP}^1$. As $c' = Sq^3 c_0$ with $c_0 = Sq^1(a^3 \cup b)$, a D-brane wrapped on B' can again be unstable.

In the last example, since $\mathbf{RP}^1$ is a circle, B' is isomorphic to $\mathbf{T}^2$. By turning on a magnetic flux on $\mathbf{T}^2$, we can endow a D-brane on B' with -1-brane charge, which takes values in $H^{10}(X;\mathbf{Z}) = \mathbf{Z}$. As there is no torsion here, a nonzero class in $H^{10}(X;\mathbf{Z})$ cannot be written as $Sq^3(\ldots)$. This example thus also makes clear that the correct statement is that *some* D-brane wrapped on B' is completely unstable, and *any* D-brane wrapped on B' can decay to a collection of -1-branes.

These have been Euclidean examples, so the unstable objects are really instantons rather than physical states of branes. For a real time example, we begin with the eight-manifold U considered at the end of section 4.1 (constructed as an $\mathbf{R}^3$ bundle over Q, or an $\mathbf{S}^3$ bundle if one prefers to compactify the fibers). We set $X = U \times \mathbf{S}^1 \times \mathbf{R}$, where $\mathbf{R}$ is the "time" direction. Then in view of the remarks in section 4.1, a threebrane wrapped on $L \times \mathbf{S}^1$ is unstable, even though $L \times \mathbf{S}^1$ is nontrivial in homology.

More generally, we can set $X = Y \times \mathbf{R}$ for any nine-dimensional spin manifold Y, with $\mathbf{R}$ still understood as the time direction. Sq^3 in general can act non-trivially on $H^3(Y;\mathbf{Z})$ or $H^4(Y;\mathbf{Z})$, but annihilates the other cohomology groups. (For example, it annihilates $H^5(Y;\mathbf{Z})$ because of (4.19).) The image of Sq^3 thus lies in $H^6(Y;\mathbf{Z})$ and $H^7(Y;\mathbf{Z})$, which equal $H_3(Y;\mathbf{Z})$ and $H_2(Y;\mathbf{Z})$. So the real time D-branes (as opposed to Euclidean signature instantons) that are destabilized by the mechanism that we have described are always two-branes or three-branes.

6. Partition Function In M-Theory

We are finally ready to analyze the partition function of the C-field on $Y = X \times \mathbf{S}^1$. Actually, we will only evaluate the contribution from C-fields that are pulled back from X – corresponding to the RR field G_4 in the Type IIA description. (The other modes would correspond in the Type IIA description to the Neveu-Schwarz B-field, which is generally omitted in the present paper.)

We treat the C-field on Y as a free field. Its modes that are pulled back from X are classified by the characteristic class $a \in H^4(X;\mathbf{Z})$. For each a there is a harmonic four-form G_a of the appropriate topological class, as in (2.4), and the kinetic energy $|G_a|^2 = \int G_a \wedge *G_a$ vanishes if and only if G_a is torsion. The partition sum we wish to evaluate is

$$\sum_{a \in H^4(X;\mathbf{Z})} (-1)^{f(a)} \exp(-|G_a|^2). \tag{6.1}$$

Here we are summing over all $a \in H^4(X; \mathbf{Z})$, while in the corresponding Type IIA expression, we would only be summing over those a's that have a lift to K-theory. As a preliminary step towards comparing (6.1) with Type IIA, we want to re-express it as a sum only over a restricted set of a's. The basic strategy for this will be to use the fact that the kinetic energy $|G_a|^2$ is invariant under $a \to a + b$ for torsion b while the M-theory phase is not. Hence, if $(-1)^{f(a)}$ vanishes upon averaging over $a \to a + b$ for a suitable set of torsion b, the contribution of a can be omitted from the partition sum. Physically, averaging over $a \to a + b$ for torsion b amounts to deriving the torsion part of a Gauss's law constraint.

6.1. An Anomaly

As a preliminary step, we first average over $a \to a + 2b$, where b is torsion. To see what this does, we first use the bilinear relation (3.13) to get

$$f(a + 2b) = f(a) + f(2b), \tag{6.2}$$

where the bilinear term can be dropped as $Sq^2(2b) = 2Sq^2 b = 0$.

Now, for $f(2b)$ we can give a simple formula, whether b is torsion or not. Note, in the context of the cobordism discussion in section 3.2, that $f(2b)$ is a cobordism invariant and so must be a linear combination of $f(b)$ and $v(b) = \int b \cup Sq^2 \lambda$. In fact, the bilinear relation gives at once

$$f(2b) = f(b) + f(b) + \int b \cup Sq^2 b = \int b \cup Sq^2 \lambda = \int b \cup w_6 = v(b), \tag{6.3}$$

where we used the fact that $2f(b) = 0$ together with (3.23). (The interested reader can use the same technique to show that $f(3b) = f(b) + v(b)$ and $f((n + 4)b) = f(nb)$.)

In view of (6.2), the partition function transforms under $a \to a+2b$ (where b is torsion) by multiplication by $(-1)^{f(2b)}$. The partition function therefore vanishes unless $f(2b) = 0$ for all torsion b. From (6.3), this means that $\int b \cup Sq^2 \lambda = 0$ for all torsion b. This integral is the torsion pairing $T(b, Sq^3 \lambda)$ described in section 3.3. Its vanishing for all torsion b is equivalent, by nondegeneracy of the torsion pairing, to

$$Sq^3 \lambda = 0. \tag{6.4}$$

If $Sq^3 \lambda \neq 0$, then the partition function vanishes. This vanishing cannot be removed by inserting local operators constructed from the C-field (as these are not sensitive to

torsion classes). We interpret it as an anomaly in the theory. An analogous anomaly was studied in [1]. The meaning of this anomaly for Type IIA will be explained in section 7.

The physical effect of a torsion C-field is precisely to give a phase to the contribution to the path integral of a wrapped brane. So, if one wishes, one could (just as in the examples studied in [1]) remove the anomaly by introducing a wrapped brane. We will explain this more fully in section 6.4 below, but for the moment we focus attention to the standard partition function.

It remains to show that the anomaly we have described is a nontrivial restriction on 10-manifolds. The argument for this is somewhat abstract and can be found in appendix D.

6.2. *Restriction On* $Sq^3(a)$

Henceforth we work on spin manifolds with $W_7 = 0$.

We want to get a restriction on the a's that contribute to the partition sum, by considering the behavior under $a \to a+c$ for c torsion. We have already imposed invariance under $a \to a + 2b$ for torsion b, so we can consider c to lie in $L = H^4_{tors}/2H^4_{tors}$, which is a vector space over the field $\mathbf{Z}_2$.

One's first thought might be that the contribution of a vanishes unless $f(a+c) = f(a)$ for all torsion c. That would be correct if $f(a+c)$ were linear in c, but it is not. By iterating the bilinear relation for f, one finds that

$$f(a+c+c') = f(a) + f(c) + f(c') + \int a \cup Sq^2(c+c') + \int c \cup Sq^2 c'. \tag{6.5}$$

The last term is the obstruction to $f(a+c)$ being linear in c.

To derive a useful constraint, we will sum over a restricted set of c's chosen so that $f(a+c)$ is a linear function on this set. For this, we simply define L' to be the subspace of L consisting of classes c such that $\int c \cup Sq^2 c' = 0$ for all torsion c'. (Since $\int c \cup Sq^2 c' = \int Sq^2 c \cup c' = T(Sq^3 c, c')$, the condition for this is that $Sq^3 c = 0$.) For $c, c' \in L'$, $f(c+c') = f(c) + f(c')$, so $f(c)$ is a linear function when restricted to L'. The bilinear relation

$$f(a+c) = f(a) + f(c) + \int c \cup Sq^2 a, \tag{6.6}$$

shows that $f(a+c)$ is likewise linear in c for $c \in L'$.

The linear function $f(c) : L' \to \mathbf{Z}_2$ can be extended (nonuniquely) to a linear function on L, and hence by the Pontryagin duality of the torsion pairing, there is a (nonunique) $P \in H^7(X;\mathbf{Z})$ with $f(c) = T(c, P)$ for all c. We can write

$$f(a + c) = f(a) + T(c, Sq^3 a + P). \tag{6.7}$$

P is unique mod the addition of an element $P' = Sq^3 c'$ for torsion c' (since these are the elements for which $T(c, P') = 0$ for all $c \in L'$).

The condition for

$$\sum_{c\in L'} (-1)^{f(a+c)} \tag{6.8}$$

to be nonzero is that $f(a+c)$ is independent of c for $c \in L'$. In other words, $T(c, Sq^3a+P) = 0$ for all such c.

Let M be the two-torsion subgroup of H^7_{tors}. L and M are vector spaces over the field $\mathbf{Z}_2$, and the torsion pairing $T : H^4_{tors} \times H^7_{tors} \to \mathbf{Z}_2 \subset U(1)$ induces a nondegenerate pairing or duality $T : L \times M \to \mathbf{Z}_2$. L' was defined as the subspace of L orthogonal to the subspace $M' = Sq^3(H^4_{tors})$ of M. We write this as $L' = (M')^\perp$. Just as in the more familiar case of linear algebra over $\mathbf{R}$, given dual vector spaces L and M and subspaces L', M' with $L' = (M')^\perp$, one has also $M' = (L')^\perp$. So the fact that $T(c, Sq^3a + P) = 0$ for all $c \in L'$ means that $Sq^3a + P \in M'$, that is, $Sq^3a + P = Sq^3c'$ for some torsion c'.

The restriction on a can thus be written

$$Sq^3a = P \bmod Sq^3(H^4_{tors}). \tag{6.9}$$

(The sign of P does not matter as P is two-torsion.)

If P is identically zero, this means that classes a that contribute to the M-theory partition function have the property that, after perhaps adding a torsion class to a, $Sq^3a = 0$. We decompose the sum over $a \in H^4(X;\mathbf{Z})$ into a sum over equivalence classes, where $a \sim a'$ if $a - a'$ is torsion. For $P = 0$, every equivalence class that contributes to the M-theory partition function has a representative that lifts to K-theory. Thus, the M-theory partition function can be written as a sum over K-theory classes. This is the expected answer from the Type IIA side – though of course we need to show that the Type IIA partition function precisely reproduces the M-theory partition function. This will be our goal in section 7.

If P is nonzero, it may be that (6.9) has no solution. Then the partition function vanishes and M-theory on $X \times \mathbf{S}^1$ is anomalous. We will eventually show in section

7.8 below that this occurs only if $W_7 \neq 0$, so it is not really a new anomaly. As a preliminary to that discussion, let us find the criterion for the existence of a solution to (6.9). If $f(c)$ is nonzero for some c such that $Sq^2 c$ can be lifted to a torsion class $d \in H^6(X;\mathbf{Z})$ – in other words, if c belongs to the space Υ introduced at the end of section 3 – then (6.9) has no solution. We prove this as follows. Suppose a obeys (6.9); using our freedom to add $Sq^3 c'$ to P for c' torsion, we can assume $Sq^3 a = P$. Then (6.9) implies (using the definition of P, as well as (6.9) and formulas from section 3) that $f(c) = T(c,P) = T(c, Sq^3 a) = \int c \cup Sq^2 a = \int Sq^2 c \cup a = \int d \cup a$. But $\int d \cup a = 0$ if d is torsion, as the cup product in integral cohomology vanishes for torsion classes. Conversely, if $f(c) = 0$ for $c \in \Upsilon$, then $f(c)$ can be regarded as a linear form on the vector space $V = H^4_{tors}/\Upsilon$ studied at the end of section 3. We showed there that the dual of V is $Sq^3(H^4(X;\mathbf{Z}))$, so that P is an element of $Sq^3(H^4(X;\mathbf{Z}))$, that is $P = Sq^3 a$ for some $a \in H^4(X;\mathbf{Z})$. In summary,

$$P \in Sq^3 H^4(X,\mathbf{Z}) \ \Leftrightarrow \ f(c) = 0 \text{ for all } c \in \Upsilon. \tag{6.10}$$

So (6.9) has no solution, rendering the M-theory anomalous, precisely if the function f is nontrivial if restricted to Υ. We interpret Υ to consist of the part of the torsion subgroup of $K(X)$ with first Chern class $c_1 = 0$. Indeed, an element $c \in \Upsilon$ has a K-theory lift because $Sq^3 c = 0$; this lift can be chosen to be an element x with Chern classes $c_1(x) = 0$, $c_2(x) = -c$, $c_3(x) = d$, and as c and d are torsion, this is compatible with x being torsion. [12]

Type IIA is anomalous if $j(x)$, defined as the mod 2 index with values in $x \otimes \overline{x}$, is nonzero for a torsion class $x \in K(X)$. In section 7, we will compare $f(c)$ to $j(x)$, and a special case of our result is that if $c \in \Upsilon$ can be lifted to a torsion class $x \in K(X)$, then

[12] To prove that x can be taken to be torsion, we must show that also c_4 and c_5 can be taken to be torsion. For this, we consider the index of the Dirac operator with values in x. It is simply $c_5(x)/4!$ So $c_5(x)$ is a multiple of 4! A bundle on $\mathbf{S}^{10}$ or equivalently a bundle that is trivial outside a small neighborhood of a point $P \in X$ can have c_5 an arbitrary multiple of 4! By adding such a bundle (and subtracting a trivial bundle of the same rank) we can set $c_5(x) = 0$ without changing $c_i(x)$ for $i < 5$. By considering the index of the Dirac operator with values in $\mathcal{L} \otimes x$, we next learn that $c_4(x)$ is a multiple of 3! If $c_4(x) = -3!\,[\Sigma]$, where $[\Sigma]$ is the class of a Riemann surface $\Sigma \subset X$, then by adding to x the K-theory class of a $D1$-brane wrapped on Σ, we can make $c_4(x)$ vanish. Since Σ is spin, this can be done with a flat Chan-Paton bundle and so without changing $c_5(x)$.

$f(c) = j(x)$ (see eq. (7.33) below). So this M-theory anomaly arises precisely if the Type IIA theory is anomalous. This is part of the matching between these two theories.

It remains to consider the possibility that P is nonzero but (6.9) has a solution. This happens if $f(c)$ is nonzero on L but annihilates the subspace Υ. In analyzing this case, we set $S = H^4(X;\mathbf{Z})/H^4_{tors}$, as at the end of section 4.2.

In this case, let a_0 be any solution of (6.9). Then the general solution is $a = a_0 + b$, where (after perhaps adding a torsion class to b) b must obey $Sq^3 b = 0$. In other words, the M-theory partition function is not written as a sum over the sublattice $S' = \ker Sq^3$ of S, which is the sublattice of S consisting of classes that have a K-theory lift. Rather, it is a sum over a coset of S' in S, namely the coset containing a_0.

To compare to Type IIA, we will have to take account of the following. The comparison to Type IIA is made not quite in terms of a but in terms of the four-form $G/2\pi$, which as we recall from (2.4) in section 2 is not quite a but $a - \lambda/2$. We want to compare the M-theory field $G/2\pi$ to a Type IIA RR field $G_4/2\pi$. The RR forms of Type IIA are defined just as differential forms, so in making this comparison, we should work mod torsion.

Let us fix a definite solution of (6.9),

$$Sq^3 a_0 = P. \tag{6.11}$$

Then we have

$$\frac{G}{2\pi} = a - \lambda/2 = b + a_0 - \lambda/2, \tag{6.12}$$

where b is an arbitrary element of S'. This formula says that $G/2\pi$ takes values in a coset of S' in $\frac{1}{2}S$, namely the coset that is generated by $a_0 - \lambda/2$.

Actually, we will need to be more precise than this. Note that $\theta_M = 2a_0 - \lambda$ is an element of S', since $Sq^3(2a_0) = 0$ and (to cancel an anomaly) we have had to assume that $Sq^3\lambda = 0$. So the allowed fluxes $G/2\pi$ in M-theory take values in a coset of S' in $\frac{1}{2}S'$, namely the coset generated by $\theta_M/2$. (This is an improved statement because, $\frac{1}{2}S'$ being a sublattice of $\frac{1}{2}S$, there are fewer cosets of S' in $\frac{1}{2}S'$ than in $\frac{1}{2}S$.)

Note that θ_M is not well-defined as an element of S', since the solution a_0 of (6.9) is not uniquely determined. But, as the ambiguity in a_0 consists of the possibility of adding to a_0 an element of S', θ_M is well-defined as an element of $S'/2S'$.

In comparing to Type IIA, we will among other things have to explain why the RR four-form flux takes values precisely in the coset of S' in $\frac{1}{2}S'$ that has just been described.

6.3. Contribution Of An Equivalence Class

We want to describe the M-theory partition function as a sum over equivalence classes of solutions of (6.9). We consider two solutions a and a' equivalent if $a - a'$ is torsion. Every equivalence class contains a representative a with $Sq^3 a = P$. The sum over the equivalence class is

$$Z_a = \exp(-|G_a|^2) \sum_{c \in H^4_{tors}} (-1)^{f(a+c)}. \tag{6.13}$$

The bilinear relation shows, given $Sq^3 a = P$, that $f(a+c) = f(a) + f(c) + T(P,c)$. So we can write

$$Z_a = \mathcal{N}(-1)^{f(a)} \exp(-|G_a|^2), \tag{6.14}$$

with

$$\mathcal{N} = \sum_{c \in H^4_{tors}} (-1)^{f(c)+T(P,c)}. \tag{6.15}$$

(6.14) expresses the contribution of an equivalence class to the partition function in terms of an overall constant $\mathcal{N}$. We want to show that $\mathcal{N} \neq 0$; indeed, vanishing of $\mathcal{N}$ would constitute a new anomaly. We will actually get a simple formula for $\mathcal{N}$, which we hope will eventually be useful in comparing the absolute normalization of the M-theory partition function to that of Type IIA (though we will not analyze all of the absolute normalization factors on the two sides in the present paper).

As we saw above, with $W_7 = 0$, the sign factor in (6.15) is invariant to $c \to c + 2c'$. So we can rewrite (6.15) as

$$\mathcal{N} = N_0 \sum_{c \in L} (-1)^{f(c)+T(P,c)} \tag{6.16}$$

where $L = H^4_{tors}/2H^4_{tors}$ and N_0 the order of the finite group $2H^4_{tors}$. The definition of P is such that $f(c) + T(P,c)$ is invariant under $c \to c + c'$ for $c' \in L' = \ker Sq^3$. So if N_1 is the order of L' and $L'' = L/L'$, we can write

$$\mathcal{N} = N_0 N_1 \sum_{c \in L''} (-1)^{g(c)}, \tag{6.17}$$

where $g(c) = f(c) + T(P,c)$.

The function $g(c)$ is quadratic:

$$g(c_1 + c_2) = g(c_1) + g(c_2) + \int c_1 \cup Sq^2 c_2. \tag{6.18}$$

The bilinear form $< c_1, c_2 >= \int c_1 \cup Sq^2 c_2$ is nondegenerate on L'' (since we have divided out its kernel, which is L'). Given our assumption that $W_7 = 0$ (and hence $f(2c) = 0$), the diagonal matrix elements of this bilinear form vanish, since (6.18) implies that $0 = g(2c) = g(c) + g(c) + \int c \cup Sq^2 c =< c, c >$. Over the field $\mathbf{Z}_2$, a quadratic form $< \ , \ >$ with vanishing diagonal matrix elements is equivalent to an antisymmetric form, and if nondegenerate, it can be block-diagonalized in 2×2 blocks in each of which it looks like

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{6.19}$$

The procedure for proving this is familiar in linear algebra over $\mathbf{R}$. We let b_1 be any element of L'' and (using the nondegeneracy) we let b_2 be any element of L'' with $< b_1, b_2 >= 1$. In the subspace generated by b_1 and b_2, the form looks like (6.19) (since the diagonal elements vanish). Repeating this procedure in the subspace of L'' orthogonal to b_1 and b_2, one gets the claimed block diagonalization.

Suppose that L'' is two-dimensional (over the field $\mathbf{Z}_2$), and so the quadratic form has precisely the shape (6.19). If we write $b = u_1 b_1 + u_2 b_2$ (with $u_1, u_2 \in \mathbf{Z}_2$), the most general quadratic function $g(b)$ obeying (6.18) on L'' is $g(b) = u_1 u_2 + \epsilon_1 u_1 + \epsilon_2 u_2$, with constants $\epsilon_1, \epsilon_2 \in \mathbf{Z}_2$. A small computation shows that in this situation

$$\sum_{b \in L''} (-1)^{g(b)} = 2(-1)^{\alpha}, \tag{6.20}$$

where $\alpha = \epsilon_1 \epsilon_2$.

Now suppose that L'' has dimension $2k$ and so the quadratic form is the sum of k blocks of the shape (6.19). The number of elements of L'' is thus $N_2 = 2^{2k}$. A function g obeying (6.17) is the sum of k functions of the sort considered in the last paragraph, one in each 2×2 block. Hence

$$\sum_{b \in L''} (-1)^{g(b)} = 2^k (-1)^{\alpha}. \tag{6.21}$$

Here $\alpha = \sum_{i=1}^{k} \alpha_i$, with α_i being defined as in the last paragraph for the i^{th} 2×2 block. α is called the "Arf invariant" of the quadratic function g. The Arf invariant is a $\mathbf{Z}_2$-valued invariant of a quadratic function g, and can be defined invariantly as the sign of the sum (6.21). Up to transformations $b \to Ab + b'$ where A is a linear transformation of L'' and $b' \in L''$, the quadratic function g is completely determined by its Arf invariant. Indeed, it is easy to show that the number of zeroes of g is $\frac{1}{2}(2^{2k} \pm 2^k)$ with the sign determined by the Arf invariant.

Our result for $\mathcal{N}$ is thus

$$\mathcal{N} = N_0 N_1 \sqrt{N_2}(-1)^\alpha = \frac{N(-1)^\alpha}{\sqrt{N_2}}, \tag{6.22}$$

where $N = N_0 N_1 N_2$ is the order of $H^4(X;\mathbf{Z})_{tors}$.

We should note that the factorization of Z_a in (6.14) depended on a specific choice of P. If we transform $P \to P + Sq^3 c_0$ (with torsion c_0), we must take $a \to a + c_0$. In this process $(-1)^{f(a)} \to (-1)^{f(a)}(-1)^{f(c_0)+T(P,c_0)}$. This sign change of $(-1)^{f(a)}$ is compensated by a change of the Arf invariant.

6.4. Comment on Brane Insertions

This paper focuses on the partition function of M-theory. Nevertheless, one is very interested, for a variety of reasons, in the computation of amplitudes with insertions of wrapped membranes. We now briefly sketch how those are formulated on an arbitrary 11-dimensional spin manifold Y. As an application we show that, while the partition function vanishes if $W_7 \neq 0$ on manifolds of the form $Y = X \times \mathbf{S}^1$, one could, if desired, always insert a torsion membrane instanton which cancels the anomaly described in section 6.1. (The rest of the paper does not depend on this construction, so the reader could omit this section.)

Let Q be the worldvolume of an M2-brane instanton. The contribution to the path integral in the presence of the C field receives a factor

$$\exp\left(i\int_Q C\right). \tag{6.23}$$

This is not a topological invariant in general when C is not flat, and a careful discussion of the instanton amplitude involves the calculus of "differential characters" [29,30]. We need not enter into such subtleties here because we are only concerned with the behavior of (6.23) under shifts of C by a *flat* field C'. A flat C-field is classified by $H^3(Y;U(1))$, and for flat C-fields, (6.23) can be regarded as the dual pairing $H_3(Y;\mathbf{Z}) \times H^3(Y;U(1)) \to U(1)$. Because this is a duality, any desired linear map from flat C-fields to $U(1)$ can be obtained as the coupling to some brane whose homology class is torsion. In particular, by picking a suitable Q, the dependence of the effective action on $C \to C + C'$ with torsion C' can be canceled. This can be done with a Q whose homology class is torsion, though the action of a brane wrapped on Q is of course positive.

7. Comparison To Type IIA

In this section, we will analyze the RR partition function in Type IIA and begin the process of demonstrating its relationship to the M-theory partition function.

7.1. Review Of The $K(X)$ Theta Function

First we recall [1,2] the general construction of a K-theory theta function, which serves as the RR partition function in Type IIA. (A precisely analogous construction based on $K^1(X)$ gives the RR partition function of Type IIB.) One starts on a ten-dimensional spin manifold X with the lattice $\Gamma = K(X)/K(X)_{tors}$. This lattice is endowed with an integer-valued unimodular antisymmetric form by the formula

$$\omega(x,y) = I(x \otimes \overline{y}), \tag{7.1}$$

where for any $z \in K(X)$, $I(z)$ is the index of the Dirac operator with values in z.[13] In any dimension of the form $4k+2$, one has $\omega(x,y) = -\omega(y,x)$. The Ramond-Ramond field $G(x)$ of a given x is defined as

$$\frac{G(x)}{2\pi} = \sqrt{\hat{A}(X)}\ \mathrm{ch}\, x, \tag{7.2}$$

with $\hat{A}$ the index density of the Dirac operator and ch the Chern character. In (7.2), we understand the right hand side to refer to the harmonic differential form in the specified real cohomology class. Note that the RR fields are defined purely as differential forms. The integral structure is defined by deriving the RR fields from an element of $K(X)$ (which has a natural integral structure, of course), not by defining integral cohomology classes associated with the RR fields.

Given a metric on X, one also endows Γ with a metric $g(x,y)$ as follows. One simply sets

$$g(x,y) = \int_X \frac{G(x)}{2\pi} \wedge \frac{*G(y)}{2\pi}, \tag{7.3}$$

where here $*$ is the Hodge duality operator. Associated with the lattice Γ is a torus $\mathbf{T} = A/\Gamma$ where A is the vector space $\Gamma \otimes_{\mathbf{Z}} \mathbf{R}$. The quantities ω and g can be interpreted

[13] This antisymmetric form is T-duality invariant. Indeed, if we exchange Type IIA with Type IIB and consider ω as being defined for D-brane states rather than for RR fields, it becomes the T-duality invariant intersection form on D-brane states introduced by Douglas and Fiol [31].

as a symplectic form and a metric, respectively, on $\mathbf{T}$. We define a complex structure J on $\mathbf{T}$ by setting

$$g(x,y) = \omega(Jx,y). \tag{7.4}$$

The metric, complex structure, and symplectic structure that we have defined turn $\mathbf{T}$ into a Kähler manifold. Suppose that $\mathcal{L}$ is a complex line bundle over $\mathbf{T}$ with positive curvature. Then $H^i(\mathbf{T};\mathcal{L}) = 0$ for $i > 0$, and according to the index theorem, the dimension $h^0(\mathcal{L})$ of $H^0(\mathbf{T};\mathcal{L})$ is

$$h^0(\mathcal{L}) = \int_{\mathbf{T}} e^{c_1(\mathcal{L})}. \tag{7.5}$$

(The Todd class, which would appear in the general index theorem for the $\overline{\partial}$ operator, is 1 for a complex torus.) Unimodularity of ω implies that

$$\int_{\mathbf{T}} e^{\omega} = 1, \tag{7.6}$$

so if we can find an $\mathcal{L}$ with $c_1(\mathcal{L}) = \omega$, then $h^0(\mathcal{L}) = 1$. In this case, $\mathcal{L}$ has, up to a constant multiple, a unique holomorphic section; this section, suitably normalized, is the RR partition function (as a function of an "external potential"). If $\mathbf{T}$ is endowed with a complex line bundle $\mathcal{L}$ with $c_1(\mathcal{L}) = \omega$, it becomes a "principally polarized abelian variety."

As was explained in detail in [28], holomorphic line bundles $\mathcal{L}$ over $\mathbf{T}$ with *constant* curvature ω are in one-one correspondence with $U(1)$-valued functions Ω on Γ such that

$$\Omega(x+y) = \Omega(x)\Omega(y)(-1)^{\omega(x,y)}. \tag{7.7}$$

(In brief, to define $\mathcal{L}$ as a unitary line bundle with connection of curvature ω, we need to specify its holonomies around noncontractible loops in $\mathbf{T}$; the role of Ω is to specify these holonomies.) While Ω cannot be taken to be identically 1, since ω is nonvanishing, one can take Ω to be valued in $\mathbf{Z}_2$. This is the case relevant to constructing the RR partition function of weakly coupled Type II superstrings.

In [1], a natural $\mathbf{Z}_2$-valued function Ω, canonically associated to a spin manifold, and obeying (7.7), was defined using the mod 2 index of the Dirac operator. The definition was as follows. For X of dimension $8k+2$, and any real vector bundle V, one has a mod 2 index $q(V)$ of the Dirac operator with values in V. More generally, for any $v \in KO(X)$, one can define the mod 2 index $q(v)$ with values in v. For any $x \in K(X)$, one has $x \otimes \overline{x} \in KO(X)$, so one can define $j(x) = q(x \otimes \overline{x})$. Then

$$\Omega(x) = (-1)^{j(x)} \tag{7.8}$$

can readily be shown [1] to obey the desired identity.

Though the identity (7.7) does not determine Ω uniquely, the formula (7.8) is distinguished because it is T-duality invariant, that is, it can be described in terms of a conformal field theory with worldsheet supersymmetry without committing oneself to a particular realization of this theory as a sigma model with a target space X. A manifestly T-duality invariant definition of $j(x)$ is as follows. Interpret x as a D-brane state in Type IIB superstring theory in the same conformal field theory background as the Type IIA model under consideration (using a different GSO projection to get IIB instead of IIA) and, in rough analogy with [31], define $j(x)$ as the number, mod 2, of zero energy states in the Ramond sector for open strings with boundary condition x at each end.[14] Though we do not have a general proof that (7.8) is the unique T-duality invariant solution of (7.7), this seems very likely.

If now $\Omega(x)$ is identically 1 for torsion elements of $K(X)$, then it can be regarded as a function on $\Gamma = K(X)/K(X)_{tors}$ and can be used to define the line bundle $\mathcal{L}$ and thence the RR partition function. If Ω is not identically 1 on $K(X)_{tors}$, then the partition function of the theory vanishes upon summing over torsion. This must be interpreted as an anomaly or inconsistency of the theory. (In [1], examples were given where nontriviality of the Ω function on torsion was related by duality to more conventional anomalies.) In our problem, we will see (in section 7.8) that the Ω function fails to be identically 1 on torsion precisely when the M-theory partition function vanishes for a similar reason.

If Ω descends to a function on Γ, we can proceed to construct a theta function that will serve as the RR partition function. To define the theta function, we pick an arbitrary splitting of Γ as a sum $\Gamma_1 \oplus \Gamma_2$, where Γ_1 and Γ_2 are "maximal Lagrangian" sublattices, that is, $\omega(x, y) = 0$ for x, y both in Γ_1 or both in Γ_2, and Γ_1 and Γ_2 are each maximal lattices with this property. It follows from this that ω establishes a duality between Γ_1 and Γ_2. An important example of the use of this duality is as follows. For $x, y \in \Gamma_2$, we have $\Omega(x + y) = \Omega(x)\Omega(y)$. Thus, Ω determines a homomorphism from Γ_2 to $\mathbf{Z}_2$. Duality of Γ_1 with Γ_2 via $\omega(\ ,\ )$ means that there exists $\theta \in \Gamma_1/2\Gamma_1$ such that

$$\Omega(y) = (-1)^{\omega(\theta,y)} \tag{7.9}$$

[14] The use of Type IIB to define the Ω function for Type IIA is admittedly slightly perplexing. Of course, to define the Ω function for Type IIB, we would similarly look at open string boundary conditions in Type IIA.

for $y \in \Gamma_2$.

The theta function is then, roughly speaking, written as a sum over Γ_1. To be more precise, it is written as a sum over a certain coset of Γ_1 in $\frac{1}{2}\Gamma_1$, namely the coset containing the element $\theta/2$, where θ was just defined. We introduce a homogeneous quadratic function τ on Γ_1, which is the period matrix of the lattice Γ with respect to its decomposition as $\Gamma_1 \oplus \Gamma_2$. (We will presently explain how to compute τ explicitly.) The theta function is then

$$\Theta = \exp(-i\pi \mathrm{Re}\ \tau(\theta/2)) \sum_{x \in \frac{1}{2}\theta + \Gamma_1} \exp\left(i\pi\tau(x)\right)\Omega(x - \theta/2). \tag{7.10}$$

The prefactor $\exp(-i\pi \mathrm{Re}\tau(\theta/2))$, which is just a constant phase multiplying the theta function, has been chosen to cancel some of the dependence of the theta function Θ on θ.[15] In fact, we have defined θ as an element of $\Gamma_1/2\Gamma_1$, but in writing the formula (7.10), θ is interpreted as an element of Γ_1. With the prefactor that we have chosen, under $\theta \to \theta + 2b$, Θ changes by an overall sign. (This can be proved using formulas we develop later.) We do not know how to fix the overall sign of the partition function, and in this paper, we will study only the dependence on the RR fields, not the overall constant normalization of the partition function. (Note that in M-theory, as we saw in section 2, to get a well-defined overall sign of the partition function requires carefully considering the fermions as well as bosons.)

In practice, as we will see, the imaginary part of $\tau(x)$ equals the conventional kinetic energy of the RR fields. The real part of τ will give an x-dependent phase factor which, together with the factor $\Omega(x - \frac{1}{2}\theta)$, must be compared to the phase factor coming from M-theory. With an obvious shift of the summation variable, we can alternatively write the theta function as

$$\Theta = \exp(-i\pi \mathrm{Re}\,\tau(\theta/2)) \sum_{x \in \Gamma_1} \exp\left(i\pi\tau(x + \theta/2)\right)\Omega(x). \tag{7.11}$$

7.2. Choice of Γ_1 *and* Γ_2

The Θ function can be written as in (7.10) for *any* Lagrangian decomposition $\Gamma = \Gamma_1 \oplus \Gamma_2$. We want to make a choice that is convenient for comparing to the conventional approach of expressing the RR partition function as a sum over $2p$-form periods for various

[15] This prefactor is also required for the partition function to be well-defined in the presence of a B-field.

p. First of all, the conventional approach is not unique, as (for example) we could treat G_0, G_2, and G_4 as the independent variables, or (by duality) one could use G_6, G_8, and G_{10}. It is both conventional and much more convenient, however, to use G_0, G_2, and G_4 as the independent variables. The reason this is convenient is that, in comparing to M-theory, we want to scale up the metric g of X by $g \to tg$ for large positive t. As we noted in the introduction, under this scaling the action $\int_X d^{10}x\sqrt{g}|G_{2p}|^2$ scales as t^{5-2p}. Hence, for large t, the representation of the partition function as a sum over fluxes of G_0, G_2, and G_4 is rapidly convergent: the nonzero fluxes are all associated with a large action. The action, in fact, is large for G_4, larger still for G_2, and largest for G_0, so we get a hierarchy of approximations: one may include G_4 only, which will be the approximation of the present section; one may include G_2 and G_4, as we do in section 9, or one may do a complete computation, as we do in section 10. If we would instead take G_6, G_8, and G_{10} as the independent variables, then as the action for these fields is small for large t, all contributions to the path integral are important, and the existence of a hierarchy of successive approximations is less apparent. Moreover, G_2 and G_4 (but regrettably not G_0) are the variables that are most easily seen in M-theory, so in comparing to M-theory it is most convenient to use a representation of the path integral in which we sum over G_2 and G_4 (and neglect G_0).

With this in mind, and with $\Gamma = K(X)/K(X)_{tors}$, there is a completely canonical choice for Γ_2: we take Γ_2 to be the subgroup of $K(X)$ consisting of classes that are torsion when restricted to the five-skeleton, modulo those that actually are torsion. Thus, a class in Γ_2 has vanishing G_0, G_2, and G_4. One might hope to take Γ_1 to be the subgroup of $K(X)$ consisting of classes with vanishing G_6, G_8, and G_{10}, but there is no such subgroup. Indeed, relations such as $c_3 = Sq^2 c_2 \bmod 2$ (for a complex vector bundle with $c_1 = 0$) make it impossible to let G_0, G_2, and G_4 vary while keeping G_6, G_8, and G_{10} zero. There is a canonical quotient Γ/Γ_2, but there is no natural way to lift this quotient to a sublattice Γ_1 of Γ.

In practice, a choice of Γ_1 gives a recipe to lift a collection of RR fields G_0, G_2, and G_4 (obeying appropriate quantization conditions) to an element of $K(X)$ mod torsion. There is no canonical choice of how to do this, but, since the theta function can be computed for any decomposition $\Gamma = \Gamma_1 \oplus \Gamma_2$, we will get the same RR partition function no matter how Γ_1 is chosen. Actually, as we will see shortly, given our choice of Γ_2, the imaginary part of τ does not depend on the choice of Γ_1, but the real part does. The factor $\Omega(x-\theta/2)$ in the

partition function also depends on the choice of Γ_1, and when both factors are included, the dependence on the choice of Γ_1 cancels out.

As an aside, it is worth noting that in some situations there are other very natural choices of lattices Γ_1, Γ_2. For example if $X = X_9 \times \mathbf{S}^1$ with a nine-manifold X_9, then $K^0(X) \cong K^0(X_9) \oplus K^1(X_9)$. One could choose $\Gamma_1 = K^0(X_9)$, $\Gamma_2 = K^1(X_9)$. In this case, it turns out that τ is imaginary and the theta function is a sum of real terms.

7.3. Computation Of τ

Here, we will carry out the explicit computation of τ. For $x \in \Gamma = K(X)/K(X)_{tors}$, we set $G(x)/2\pi = \sqrt{\hat{A}}\,\mathrm{ch}\,(x)$. The metric and symplectic form on Γ are defined, as we explained above, by

$$\begin{aligned} g(x,y) &= \frac{1}{(2\pi)^2}\int_X G(x)\wedge *G(y) \\ \omega(x,y) &= \frac{1}{(2\pi)^2}\int_X G(x)\wedge G(\overline{y}) = -\frac{1}{(2\pi)^2}\sum_{q=0}^{5}(-1)^q\int_X G_{2q}(x)\wedge G_{10-2q}(y). \end{aligned} \tag{7.12}$$

In the last step, we use the fact that $y \to \overline{y}$ acts on the RR fields by $G_{2p} \to (-1)^p G_{2p}$. The complex structure J on $\Gamma \otimes_{\mathbf{Z}} \mathbf{R}$ is defined by

$$\omega(Jx,y) = g(x,y). \tag{7.13}$$

Explicitly, this means that

$$(-1)^{p+1}G_{2p}(Jx) = *(G_{10-2p}(x)). \tag{7.14}$$

As above, we let Γ_2 be the sublattice of Γ with $G_0 = G_2 = G_4 = 0$ (corresponding to K-theory elements whose restriction to the five-skeleton in X is torsion), and we let Γ_1 be any complementary Lagrangian sublattice. We pick a basis y_i of Γ_2 and a dual basis x^i of Γ_1:

$$\omega(x^i, y_j) = \delta^i{}_j, \quad \omega(x^i, x^j) = \omega(y_i, y_j) = 0. \tag{7.15}$$

Explicitly evaluating $\omega(x^i, y_j)$ from the definition of ω and the fact that $y_j \in \Gamma_2$, we have

$$\delta^i{}_j = \frac{1}{(2\pi)^2}\int_X \left(-G_4(x^i)G_6(y_j) + G_2(x^i)G_8(y_j) - G_0(x^i)G_{10}(y_j)\right). \tag{7.16}$$

The period matrix $\tau(x^i,x^j)$, also denoted τ^{ij}, is defined by requiring that

$$Z^i = x^i + \sum_j \tau^{ij} y_j \tag{7.17}$$

should obey $J(Z^i)=\sqrt{-1}Z^i$ for all i, where J is extended to act complex-linearly. Similarly, extending G to act complex-linearly, can can use (7.14) to obtain:

$$\sqrt{-1}(-1)^{p+1}G_{2p}(x^i+\sum_j \tau^{ij}y_j) = *\left(G_{10-2p}(x^i+\sum_j \tau^{ij}y_j)\right). \tag{7.18}$$

Setting $10-2p=2q$ and using $G_{2q}(y_j)=0$ for $q=0,1,2$, we get

$$(-1)^{q+1}G_{10-2q}(x^i)+(-1)^{q+1}\sum_j \tau^{ij}G_{10-2q}(y_j) = \sqrt{-1}*(G_{2q}(x^i)), \quad q=0,1,2. \tag{7.19}$$

If one takes the cup product of this formula with $G_{2q}(x^k)$ and sums over $q=0,1,2$, one gets a formula for τ^{ij}:

$$\tau^{ij} = \sqrt{-1}\sum_{q=0,1,2}\int_X \frac{G_{2q}(x^i)}{2\pi}\wedge\frac{*G_{2q}(x^j)}{2\pi} + \sum_{q=0,1,2}(-1)^q\int_X \frac{G_{10-2q}(x^i)}{2\pi}\wedge\frac{G_{2q}(x^j)}{2\pi}. \tag{7.20}$$

$\mathrm{Im}(\tau^{ij})$ is manifestly symmetric in i and j; to prove symmetry of $\mathrm{Re}(\tau^{ij})$, one uses $\omega(x^i,x^j)=0$.

Any $x\in\Gamma_1$ has an expansion $x=\sum_i f_i x^i$ with integers f_i. We define $\tau(x) = \sum_{ij} f_i f_j \tau(x^i,x^j)$. $\mathrm{Im}\,\tau(x)$ is the conventional kinetic energy of the RR fields G_0, G_2, and G_4 associated with x. For generic x, $G(x)$ also has nonzero components G_{2p} for $p\geq 3$; these depend on the non-canonical choice of Γ_1, but do not appear in $\mathrm{Im}\,\tau$. On the other hand, $\mathrm{Re}\,\tau$ is a topological invariant (independent of the metric on X), but does depend on the higher components of $G(x)$. We will show in section 7.4 that the Θ function is independent of the choice of Γ_1. To do this, it is useful to note that while the function $\tau(x)$ is initially only defined for $x\in\Gamma_1$, the explicit formula (7.20) makes sense for all $x\in\Gamma$. The resulting extension of $\tau(x)$ has the the nice property that for any $x\in\Gamma$ and for any $y\in\Gamma_2$, one has

$$\begin{aligned}\mathrm{Im}\,\tau(x+y) &= \mathrm{Im}\,\tau(x)\\ \mathrm{Re}\,\tau(x+y) &= \mathrm{Re}\,\tau(x)+\omega(x,y).\end{aligned} \tag{7.21}$$

7.4. Existence Of Description Via G_0, G_2, And G_4

Naively speaking, the RR partition function in Type IIA superstring theory is defined as a sum over G_0, G_2, and G_4 fields with certain quantization conditions on the periods. We will now show that such a description does hold in Type IIA superstring theory, but that both the quantization conditions on the G_{2p} and the phases with which different terms contribute to the path integral are unusual. Since the theta function is written as a sum over Γ_1, the quantization condition on the G_{2p} is that there must exist $x \in \Gamma_1$ such that

$$\frac{G_{2p}}{2\pi} = \left(\sqrt{\hat{A}}\ \mathrm{ch}(x+\theta/2)\right)_{2p} \text{ for } p = 0, 1, 2. \tag{7.22}$$

We will now show that the values of G_{2p} allowed by this relation for $p = 0, 1, 2$ are independent of the choice of Γ_1. This can be seen as follows. Any change of Γ_1, keeping Γ_2 fixed, can be implemented by selecting a map $f : \Gamma_1 \to \Gamma_2$, obeying

$$\omega(x_1, f(x_2)) + \omega(f(x_1), x_2) = 0, \text{ for } x_1, x_2 \in \Gamma_1. \tag{7.23}$$

Given such a map, one replaces Γ_1 by the Lagrangian lattice $\hat{\Gamma}_1$ that consists of elements $\hat{x} = x + f(x)$ for $x \in \Gamma_1$. Since $G_{2p}(f(x)) = 0$ for $p = 0, 1, 2$, the condition (7.22) is not affected by this transformation. Similarly, in changing lattices, θ is mapped to $\hat{\theta} = \theta + f(\theta)$ (a transformation that preserves the defining property of θ, namely that $(-1)^{\omega(\theta,y)} = \Omega(y)$ for $y \in \Gamma_2$). This likewise does not modify the condition (7.22).

Having found the quantization conditions on the G_{2p}, can one forget the rest of the K-theory formalism? Not quite. Naively, one would expect to weight a given set of RR fields by the exponential of the classical supergravity action. This exponential is positive (as long as the NS B-field, which would produce a phase, vanishes). However, the K-theory formalism gives us a phase. Given G_0, G_2, and G_4 which are correctly quantized – so that a solution x of (7.22) exists – we pick such a solution, which is contained in Γ_1 for some choice of Γ_1, and then we discover from (7.11) that the contribution of x to the partition function is

$$Z_x = \exp(-i\pi \mathrm{Re}\ \tau(\theta/2)) \exp(i\pi\tau(x+\theta/2))\Omega(x). \tag{7.24}$$

Let us now verify that the contribution to the partition function of a given G_0, G_2, and G_4, depends only on those fields and not on the choice of x. For this, we must show that Z_x as defined in (7.24) is invariant under

$$x \to x + f(x),\ \theta \to \theta + f(\theta), \tag{7.25}$$

for any linear map $f : \Gamma_1 \to \Gamma_2$ that obeys (7.23). To prove this, we must recall the multiplicative property of Ω:

$$\Omega(x + f(x)) = \Omega(x)\Omega(f(x))(-1)^{\omega(x,f(x))} = \Omega(x)(-1)^{\omega(x+\theta,f(x))}. \tag{7.26}$$

In the second step, we have used the fact that $\Omega(f(x)) = (-1)^{\omega(\theta,f(x))}$, since $f(x) \in \Gamma_2$. In addition, we must use (7.21) and (7.23) to show that

$$\tau(x+\theta/2+f(x)+f(\theta/2))-\tau(x+\theta/2) = \omega(x,f(x))+\omega(\theta,f(x))+\omega(\theta/2,f(\theta/2)). \tag{7.27}$$

The last term cancels the transformation law of the prefactor in (7.11), and putting the pieces together, we learn that Z_x indeed has the claimed symmetry (7.24).

Thus, as one would naively expect, the RR partition function in Type IIA can be written as a sum over a certain lattice of allowed values of G_{2p} fluxes for $p = 0, 1, 2$, with a precise recipe for the contribution of each lattice point. The formula, however, is surprisingly subtle. The phase factor coming from $\mathrm{Re}\,\tau$ depends on cohomology operations such as Sq^2 (which constrains G_6 in terms of the G_{2p} with $p \leq 2$). In addition, there is a sign factor $\Omega(x)$ coming from the mod 2 index; this factor is even more subtle, in that there is no cohomological formula for the mod 2 index, even using operations such as Sq^2. The usual Type IIA supergravity Lagrangian misses the phase factor in Z_x, and this is quite natural since those factors really cannot be described using conventional ingredients.

"Why" does the RR partition function of Type IIA contain such subtle phase factors? One way to explain this is via T-duality. Starting with a description of the partition function as a sum over G_0, G_2, and G_4 fluxes only, T-duality will mix in higher RR fields. To return to a description via G_{2p} for $p \leq 2$, one will have to accompany a T-duality transformation with a spacetime duality. The spacetime duality will generate phases. There is consequently no T-duality invariant partition function without phases. A manifestly T-duality invariant construction is the K-theory theta function. When reduced to a description via G_0, G_2, and G_4, it takes the form that we have described.

Now that we have established the existence in Type IIA of a description via G_0, G_2, and G_4, our main goal in the rest of this paper will be to compare this description to what comes from M-theory. In fact, in the present section, we consider only the contributions with $G_0 = G_2 = 0$. After working out the M-theory phase on circle bundles in section 8 we will make a comparison including G_2 in section 9.

7.5. Comparing E_8 and K-theory mod two indices

One key ingredient in comparing the M-theory and K-theory theta functions is the relation between the mod two indices used to define these two functions. Accordingly, let us consider a class $a \in H^4(X,Z)$ which has a K-theory lift x. As we have seen in section 3.1, we can assume there is a rank 5 $SU(5)$ bundle E with $x = E - F$ where F is a trivial rank 5 bundle. Thus, $c_1(x) = 0, c_2(x) = -a$. We have then $x \otimes \overline{x} = E \otimes \overline{E} \oplus F \otimes \overline{F} - E \otimes \overline{F} - \overline{E} \otimes F$. $E \otimes \overline{E}$ is the same as $\mathrm{ad}(E) \oplus \mathcal{O}$, where $\mathcal{O}$ is a trivial line bundle and $\mathrm{ad}(E)$ is the bundle derived from E in the adjoint representation of $SU(5)$. $F \otimes \overline{F}$ is the same as 25 copies of $\mathcal{O}$. So for purposes of mod 2 index theory, we can replace $E \otimes \overline{E} \oplus F \otimes \overline{F}$ by $\mathrm{ad}(E)$. Likewise as F is a trivial bundle of odd rank, we can replace $E \otimes \overline{F} \oplus \overline{E} \otimes F$ by $E \oplus \overline{E}$, and the mod 2 index with values in this bundle is the mod 2 reduction of $I(E)$, the ordinary index with values in E. So

$$\Omega(x) = (-1)^{q(\mathrm{ad}(E)) + I(E)}, \tag{7.28}$$

where we recall that q denotes the mod 2 index.

Now let us compare with the M-theory phase. For this, we simply construct an E_8 bundle $V(a)$ in the adjoint representation with characteristic class a. We can then relate $V(a)$ to E using the familiar embedding of $SU(5) \times SU(5) \subset E_8$. The decomposition of the adjoint representation of E_8 was given in equation (3.27). Roughly as in that discussion, we take the first $SU(5)$ bundle to be E, and the second to be the trivial bundle F. Then, discarding representations that appear with even multiplicity, the adjoint E_8 bundle can be expressed in terms of E as $\mathrm{ad}(E) \oplus \wedge^2 E \oplus \wedge^2 \overline{E}$. The mod 2 index with values in this bundle is $q(\mathrm{ad}(E)) + I(\wedge^2 E)$. So

$$f(a) = q(\mathrm{ad}(E)) + I(\wedge^2 E). \tag{7.29}$$

Now we can compare (7.28) to (7.29) using the index theorem. If we define $\mathrm{ch}_t(x) = \sum t^k \mathrm{ch}_k(x)$ for any class x, then we can use the splitting principle to derive:

$$\mathrm{ch}_t(\Lambda^2 E) = \frac{1}{2}(\mathrm{ch}_t(E))^2 - \frac{1}{2}\sum_{k \geq 0}(2t)^k \mathrm{ch}_k(E) \tag{7.30}$$

In particular, $\mathrm{ch}_5(\Lambda^2(E)) = (\mathrm{ch}_2\mathrm{ch}_3 + \mathrm{ch}_1\mathrm{ch}_4 - 11\mathrm{ch}_5)(E)$, so we get index densities

$$\begin{aligned} i(E) &= \frac{c_5(E) - (c_2(E) + \lambda)c_3(E)}{24} \\ i(\Lambda^2(E)) &= \frac{-11c_5(E) - (c_2(E) + \lambda)c_3(E)}{24} \end{aligned} \tag{7.31}$$

and it follows that for any $SU(5)$ bundle E,

$$I(E) + I(\wedge^2 E) = \int_X \frac{\lambda c_3(E) + c_2(E)c_3(E)}{2} \mod 2. \tag{7.32}$$

(Note that it follows from (7.31) that $c_5(E)/2$ is integral, and moreover that $c_5(E) - (c_2(E) + \lambda)c_3(E)$ is divisible by 24, and hence that $\frac{1}{2}(\lambda + c_2(E))c_3(E)$ is integral.)

Putting these equations together, we obtain the following key result. If a has a K-theory lift x, then

$$(-1)^{f(a)} = \Omega(x) e^{\frac{i\pi}{2}\int(\lambda + c_2(x))c_3(x)} \tag{7.33}$$

It follows, in particular, that the right hand side of (7.33) is independent of the lift x of a. To compare M-theory to Type IIA, we still need a knowledge of the "characteristic."

7.6. Evaluation Of The Characteristic

One important ingredient in the Type IIA theta function is the "characteristic" $\theta \in \Gamma_1/2\Gamma_1$, defined by the condition $(-1)^{\omega(\theta,y)} = \Omega(y)$ for $y \in \Gamma_2$. We will here compute θ, which can be regarded as an element of $\Gamma/(2\Gamma \oplus \Gamma_2)$, which is the same as $\Gamma_1/2\Gamma_1$. We will show that

$$G_0(\theta) = G_2(\theta) = 0. \tag{7.34}$$

Then we will show that

$$\frac{G_4(\theta)}{2\pi} = -\lambda + 2a_0 \mod 2 \ker Sq^3, \tag{7.35}$$

where a_0 is a class encountered on the M-theory side in section 6. $G_4(\theta)/2\pi$ is only determined modulo $2\ker Sq^3$ simply because θ is only uniquely defined mod 2Γ; adding to θ an element of 2Γ that is trivial on the three-skeleton will add an element of $2\ker Sq^3$ to $G_4(\theta)/2\pi$. (7.34) and (7.35) are approximations to the following more precise description of θ: θ is trivial on the three-skeleton of X, and its image in the Atiyah-Hirzebruch spectral sequence (see appendix C for more detail) is the class $-\lambda + 2a_0$ given in (7.35). This uniquely determines θ modulo the possibility of adding a K-theory class trivial on the five-skeleton, that is, an element of Γ_2. So the above description completely characterizes θ as an element of $\Gamma/(2\Gamma + \Gamma_2)$.

To verify the above properties of θ, we will proceed as far as we can with a direct, elementary computation. This will be done by representing K-theory classes in terms of branes with even-dimensional world-volume. Such branes enter more directly in the physics

of Type IIB superstring theory, but here we will use them in computing the Ω function of Type IIA. (A similar technique was used in section 7.1 to demonstrate the T-duality invariance of Ω.)

The basic tool in the direct computation will be a fact explained in section 4 of [2]. If a K-theory class y can be represented by a D-brane wrapped on a submanifold Q_y of spacetime (and endowed with some Spin^c structure), then the mod 2 index $j(y)$ with values in $y \otimes \overline{y}$ is equal to $\nu(Q_y)$, the number mod 2 of zero modes of the worldvolume fermions of the brane wrapped on Q_y. (The worldvolume fermions are spinors of Q_y with values in spinors of the normal bundle to Q_y, subject to the usual chirality projection.) Then, since $\Omega(y)$ is defined as $(-1)^{j(y)}$, we get

$$\Omega(y) = (-1)^{\nu(Q_y)}. \tag{7.36}$$

θ, therefore, is characterized by

$$\omega(\theta, y) = \nu(Q_y) \bmod 2 \tag{7.37}$$

for $y \in \Gamma_2$.

To detect $G_0(\theta)$, we take y to have $G_{2p}(y) = 0$ except for $p = 5$. This means that Q_y should be a -1-brane or a point p in X. The Dirac operator of a point is zero; it acts in this case on a rank 16 bundle (the spinors of the normal bundle), so the number of zero modes is 16. So $\omega(\theta, y) = 0$ mod 2 if y is dual to a point, and we can pick θ to be trivial up to the two-skeleton of X.

To evaluate θ on the two-skeleton, we must evaluate (θ, y) where y is dual to a Riemann surface Σ in X (so that $G_{2p}(y) = 0$ for $p < 4$). As X and Σ are spin, the normal bundle to Σ in X is spin. A spin bundle on a Riemann surface is trivial, so the normal bundle is a trivial rank eight bundle. The positive or negative chirality spinors of the normal bundle are hence trivial rank eight bundles, and the number of zero modes of the Dirac operator of the world-volume fermions is divisible by eight. Hence, $\omega(\theta, y) = 0$ if y is dual to a Riemann surface. So we can pick θ to be trivial up to the four-skeleton of X.

To evaluate θ on the four-skeleton, we must evaluate (θ, y) where $G_{2p}(y) = 0$ for $p < 3$. Any y that is the K-theory class of a four-manifold Q_y has this property (but as we explain later, there are additional y's, so the direct computation we are about to make will not give a complete answer). For such y's, since $G_0(\theta) = G_2(\theta)$, we have $(\theta, y) = \int_{Q_y} G_4(\theta)/2\pi$.

If we evaluate this expression using (7.35), we find that (mod 2), the $2a_0$ term does not contribute, and that (7.35) implies

$$\nu(Q_y) = (\theta, y) = \int_{Q_y} \lambda \bmod 2. \tag{7.38}$$

This formula for $\nu(Q_y)$ is correct; it can be deduced by using index theory to count the fermion zero modes on Q_y, as in [4]. (The minus sign in (7.35) is a choice made for convenience in comparison to M-theory.)

The reason that this computation does not completely determine θ on the four-skeleton is that the condition $G_{2p}(y) = 0$ for $p < 3$ does not imply that y is dual to a four-manifold. It implies that y is torsion on the five-skeleton of X, but y must actually be trivial on the five-skeleton to be the K-theory class of a four-manifold (which has codimension six in X). It is perhaps helpful to recall that the class a_0 entered in section 6 in considering the M-theory contribution of classes $a \in H^4(X; \mathbf{Z})$ that are torsion, and can be lifted to K-theory, but whose K-theory lift y cannot be chosen to be torsion.[16] In this situation, y is trivial on the three-skeleton of X, $c_2(y) = -a$ and y is torsion on the five-skeleton, but y is not torsion on the six-skeleton. To completely determine θ on the four-skeleton, we need to consider the pairing of θ with such classes y.

There is no way to make this comparison using elementary formulas, since the definition of a_0 in section 6 involved a mod 2 index for which there is no explicit formula. To proceed with branes, we would have to represent y as the K-theory class of a fivebrane, in which case we would meet the mod 2 index of the worldvolume fermions in six dimensions and would have to relate this to the E_8 mod 2 index considered in section 6.

Instead of proceeding precisely in this fashion, we will take for our starting point the formula (7.33) derived above. We will apply this to our problem of deriving the characteristic θ by taking a to be a torsion class that can be lifted to a K-theory class $y = E - F$ (where F is trivial of the same rank as E, $c_1(E) = 0$, and $c_2(E) = -a$). The E_8 mod 2 index $f(a)$ is

$$f(a) = \int_X Sq^2 a_0 \cup a. \tag{7.39}$$

This was the definition (6.11) of a_0. In the present case, $\int c_2(E)c_3(E) = 0$ as $c_2(E)$ is torsion. We also have $\int_X Sq^2 a_0 \cup a = \int_X a_0 \cup Sq^2 a = \int_X a_0 \cup c_3(E)$ mod 2 (where we recall

[16] There is no subtlety analogous to this in the cases considered above, because there is no torsion in $H^0(X; \mathbf{Z})$, and torsion in $H^2(X; \mathbf{Z})$ can always be lifted to torsion in K-theory by finding a suitable line bundle.

that $c_3(E) = Sq^2 c_2(E)$ mod 2). And we can identify $c_3(E)$ as $c_3(y)$. We have then from (7.33)

$$\Omega(y) = (-1)^{\frac{1}{2}\int_X (\lambda - 2a_0)\cup c_3(y)}. \tag{7.40}$$

(Since the exponent is only defined mod 2, we can make choices of signs. These are chosen for convenience in comparing to M-theory.)

Now, we have shown above that θ has the property that $G_{2p}(\theta) = 0$ for $p = 0, 1$. Given this, it follows that for y as in the last paragraph (so that in particular $G_{2p}(y) = 0$ for $p = 0, 1, 2$), we have

$$\omega(\theta, y) = -\int_X \frac{G_4(\theta)}{2\pi} \wedge \frac{G_6(y)}{2\pi} = -\frac{1}{2}\int_X \frac{G_4(\theta)}{2\pi} \wedge c_3(y). \tag{7.41}$$

Comparing the last two formulas, we see that to achieve $\Omega(y) = (-1)^{\omega(\theta,y)}$ for such y's, we need the result that was claimed in (7.35) for $G_4(\theta)/2\pi$.

This result has a significance that has already been explained in the discussion of eqn. (6.12). In section 6, we learned that the M-theory partition function can be written as a sum over certain equivalence classes. Once we pick a solution a_0 of $Sq^3 a_0 = P$, each equivalence class contains a representative $a = a_0 + b$, where $Sq^3 b = 0$. The M-theory four-form is

$$\frac{G}{2\pi} = -\frac{\lambda}{2} + a = \frac{-\lambda + 2a_0}{2} + b. \tag{7.42}$$

Here b can be lifted to K-theory as an element $x(b) \in \Gamma_1$. In Type IIA, G is interpreted as the RR form G_4 (with an additional correction once we turn on G_2, as we do in the next section), and in view of our result for θ, (7.42) is equivalent to the standard Type IIA formula

$$\frac{G_4}{2\pi} = \left(\sqrt{\hat{A}}\,\mathrm{ch}(\theta/2 + x(b))\right)_4. \tag{7.43}$$

The M-theory sum over b corresponds in Type IIA to the sum over the coset of Γ_1 in $\frac{1}{2}\Gamma_1$ that is generated by $\theta/2$.

7.7. Comparison Of Phases

As found in section 6, the phase of the contribution of a given equivalence class to the M-theory partition function is $(-1)^\alpha(-1)^{f(a_0+b)}$, where α is the Arf invariant of a certain quadratic function. By the bilinear relation this is

$$(-1)^{\alpha + f(a_0)}(-1)^{f(b) + \int a_0 \cup Sq^2 b}. \tag{7.44}$$

The factor $(-1)^{\alpha+f(a_0)}$ is independent of a_0, and, of course, also independent of b. In the present paper, we will not try to understand the absolute normalization of the M-theory and Type IIA partition functions, but only the dependence on RR fields. Up to a constant factor, the sign of the contribution to the path integral of an equivalence class with a representative $a = a_0 + b$ is

$$\varphi_M(b) = (-1)^{f(b)+\int a_0 \cup Sq^2 b}. \tag{7.45}$$

Using (7.29) above, if E is an $SU(5)$ bundle with $c_2(E) = -b$, we can write this as

$$\varphi_M(b) = (-1)^{q(\mathrm{ad}(E))+I(\wedge^2 E)+\int a_0 \cup c_3(E)}. \tag{7.46}$$

We want to compare this to the corresponding phase on the Type IIA side. This is

$$\varphi_{IIA}(b) = \exp(-i\pi \mathrm{Re}\tau(\theta/2)) \exp(i\pi \mathrm{Re}\tau(x+\theta/2))\Omega(x), \tag{7.47}$$

where $x = x(b)$ is the K-theory class $E - F$, F being a trivial rank five bundle. With the help of (7.28), this becomes

$$\varphi_{IIA}(b) = \exp(-i\pi \mathrm{Re}\tau(\theta/2)) \exp(i\pi \mathrm{Re}\tau(x+\theta/2))(-1)^{q(\mathrm{ad}(E))+I(E)}. \tag{7.48}$$

Finally, we must evaluate $w = \mathrm{Re}\,\tau(x+\theta/2) - \mathrm{Re}\,\tau(\theta/2)$. This is given by

$$\begin{aligned} w &= -\int \left(\frac{G_4(x)}{2\pi} + \frac{1}{2}\frac{G_4(\theta)}{2\pi}\right) \wedge \left(\frac{G_6(x)}{2\pi} + \frac{1}{2}\frac{G_6(\theta)}{2\pi}\right) + \frac{1}{4}\int \frac{G_4(\theta)}{2\pi} \wedge \frac{G_6(\theta)}{2\pi} \\ &= -\int \frac{G_4(x)}{2\pi} \wedge \frac{G_6(x)}{2\pi} - \int \frac{G_4(\theta)}{2\pi} \wedge \frac{G_6(x)}{2\pi} \\ &= \frac{1}{2}\int c_2(E)c_3(E) + \frac{1}{2}\int (\lambda - 2a_0)c_3(E) \end{aligned} \tag{7.49}$$

With (7.46) and (7.48) as well as the last formula, we get

$$\varphi_M(b) = \varphi_{IIA}(b)(-1)^{I(E)+I(\wedge^2 E)-\frac{1}{2}\int_X (c_2(E)+\lambda)c_3(E)} = \varphi_{IIA}(b), \tag{7.50}$$

where in the last step, (7.32) has been used.

This completes the proof that the M-theory sum over G-fields reproduces the Type IIA sum over fluxes of the RR four-form, whenever the anomalies cancel on both sides. To complete the picture, we will now show that the anomaly cancellation condition is also the same on the two sides.

7.8. Criterion For Anomaly

In M-theory, we found at several points that the theory is anomalous unless the spin manifold X has $W_7 = 0$. In Type IIA, we have found only one possibility of an anomaly: the theory is anomalous if the $\mathbf{Z}_2$-valued function Ω on $K(X)$ is nontrivial when restricted to torsion classes. For then, the partition function vanishes when summed over torsion, and the vanishing cannot be lifted by any local observable.

So to match the two theories, we hope to show that Ω vanishes on torsion classes if and only if $W_7 = 0$. The main step is to repeat the analysis of the class θ presented in section 7.3 without assuming that the anomaly cancels.

First, when restricted to classes that are torsion on the five-skeleton of X, Ω is a homomorphism to $\mathbf{Z}_2$; that is, on such classes, it obeys $\Omega(y_1 + y_2) = \Omega(y_1)\Omega(y_2)$, as $\omega(y_1, y_2) = 0$. Ω can be extended, though not canonically, to a homomorphism $F : K(X) \to \mathbf{Z}_2$.

We must recall Poincaré duality in K-theory, which asserts that there is a Pontraygin duality

$$K(X) \times K(X; U(1)) \to U(1). \tag{7.51}$$

This means that for any $x \in K(X)$, $y \in K(X; U(1))$, there is a $U(1)$-valued pairing (x, y), linear in each variable, such that any homomorphism $F : K(X) \to U(1)$ is $x \to (x, f)$ for some $f \in K(X; U(1))$. Applying this to our homomorphism $F : K(X) \to \mathbf{Z}_2 \subset U(1)$, we conclude that $F(x) = (x, \overline{\theta})$ for some $\overline{\theta} \in K(X; U(1))$. Since F maps to $\mathbf{Z}_2$, $\overline{\theta}$ can actually be regarded as an element of $K(X; \mathbf{Z}_2)$.

Now from the exact coefficient sequence $0 \to \mathbf{Z} \xrightarrow{2} \mathbf{Z} \xrightarrow{r} \mathbf{Z}_2 \to 0$ (where the first map is multiplication by 2 and the second is mod 2 reduction), we get a K-theory exact sequence

$$\cdots \to K(X) \xrightarrow{r} K(X; \mathbf{Z}_2) \xrightarrow{\delta} K^1(X) \to \cdots. \tag{7.52}$$

Here δ is the "connecting homomorphism," analogous to the Bockstein map in cohomology.

For y and z torsion classes in $K(X)$ and $K^1(X)$, one defines a torsion pairing $T_K(y, z)$ analogous to the torsion pairing in cohomology that was introduced in section 4.2. In fact, $T_K(y, z) = (y, w)$, where $w \in K(X; U(1))$ is such that $\delta(w) = z$. [17] For example,

[17] δ is the connecting homomorphism $\delta : K(X; U(1)) \to K^1(X)$ associated with a long exact sequence like (7.52) derived from the coefficient sequence $0 \to \mathbf{Z} \to \mathbf{R} \to U(1) \to 0$. We really only need the $\mathbf{Z}_2$ case.

$T_K(y, \delta(\overline{\theta})) = (y, \overline{\theta})$. T_K is nondegenerate just like the torsion pairing in cohomology. Thus, there is a torsion class y with $T_K(y, \delta(\overline{\theta})) \neq 0$ if and only if $\delta(\overline{\theta}) \neq 0$. Since $T_K(y, \delta(\overline{\theta})) = (y, \overline{\theta}) = F(y)$, this says that $F(y)$ vanishes on torsion classes, and thus Type IIA is anomaly-free, if and only if $\delta(\overline{\theta}) = 0$.

On the other hand, from exactness of (7.52), vanishing of $\delta(\overline{\theta})$ is precisely the condition for being able to lift $\overline{\theta}$ to a class $\theta' \in K(X)$ that reduces to $\overline{\theta}$ mod 2. We can calculate the condition for this in another way.

First we make a remark that holds whether $\overline{\theta}$ can be lifted or not. The restriction of $\overline{\theta}$ to the five-skeleton is completely determined by the fact that $(y, \overline{\theta}) = \Omega(y)$ whenever y is trivial on the five-skeleton. Proceeding exactly as in the proof of (7.34) and (7.35), one can show that $\overline{\theta}$ is trivial on the three-skeleton and that the obstruction to trivializing it on the four-skeleton is the class w_4 which is the mod 2 reduction of λ. Existence of the class $\overline{\theta} \in K(X; \mathbf{Z}_2)$ whose image in the Atiyah-Hirzebruch spectral sequence for $K(X; \mathbf{Z}_2)$ is w_4 means that the differentials in the AHSS annihilate w_4. The first such differential is $d_3' = Sq^2 Sq^1 + Sq^1 Sq^2$, regarded as a map on the $\mathbf{Z}_2$ cohomology. Since w_4 is the reduction of an integral class λ, we have $d_3' w_4 = Sq^1 Sq^2 w_4 = Sq^3 w_4 = w_7$. We conclude that $w_7 = 0$ for ten-dimensional spin manifolds.

Now suppose that $\overline{\theta}$ can be lifted to a class $\theta' \in K(X)$. Then θ' is divisible by 2 on the three-skeleton, since $\overline{\theta}$ vanishes there, and (by adding to θ' two copies of a suitable sum of line bundles) one can assume that θ' is trivial on the three-skeleton. On the four-skeleton, θ' is measured by a cohomology class that reduces mod 2 to w_4. Any such class is $-\lambda + 2a_0$ for some a_0. It follows that the class $-\lambda + 2a_0$, for some a_0, is annihilated by the differentials in the Atiyah-Hirzebruch spectral sequence for $K(X)$. The first such differential is Sq^3, regarded now as a map on the integral cohomology, so we have $0 = Sq^3(\lambda - 2a_0) = Sq^3 \lambda = W_7$. Thus, $W_7 = 0$ precisely when $\overline{\theta}$ can be lifted to a class $\theta' \in K(X)$, or in other words precisely when Type IIA is anomaly-free.

When θ' exists, it is in fact precisely the "characteristic" that we have called θ in defining the Type IIA theta function.

We can now close a gap left open in the discussion of (6.10), and show that $W_7 = 0$ indeed implies that (6.9) always has solutions. Indeed, suppose $c \in \Upsilon$. Then $Sq^3(c) = 0$ so c has a K-theory lift $x(c)$. Moreover, since $c \in \Upsilon$, one may choose the class $x(c)$ to be torsion. In this case, by (7.33), $f(c) = j(x(c))$. However, we have just seen that when $W_7 = 0$ we have $j(x(c)) = 0$. Therefore, $f(c) = 0$ and so by (6.10) $Sq^3(a) = P$ has a solution.

8. Including G_2 In M-Theory

8.1. Evaluation Of The η Invariant

So far we have evaluated the phase of the M-theory effective action, described in section 2 in terms of E_8 gauge theory, only for eleven-manifolds of the form $Y = X \times \mathbf{S}^1$. Now we are going to generalize the discussion to consider the case that Y is an $\mathbf{S}^1$ bundle over X. We assume that the metric on Y is invariant under rotations of the $\mathbf{S}^1$ fibers, and that the C-field on Y, and hence the E_8 bundle, is pulled back from X. (We will later add to C a topologically trivial term that is not a pullback.) Also, we continue to assume that the spin structure on $\mathbf{S}^1$ is supersymmetric (unbounding).

The $\mathbf{S}^1$ bundle $Y \to X$ is the bundle of unit vectors in a complex line bundle $\mathcal{L}$. The basic idea will be to calculate by Fourier transforming in the $\mathbf{S}^1$ direction. Consider functions on Y that transform as $e^{-ik\theta}$ under rotations of the $\mathbf{S}^1$, for some integer k. In their X-dependence, they can be interpreted as sections of $\mathcal{L}^k$. Thus we have a decomposition

$$\mathrm{Fun}(Y) = \oplus_{k\in\mathbf{Z}}\Gamma(X, \mathcal{L}^k). \tag{8.1}$$

Here $\mathrm{Fun}(Y)$ is the space of functions on Y, and $\Gamma(X, \mathcal{L}^k)$ the space of sections of $\mathcal{L}^k$.

Consider an $\mathbf{S}^1$-invariant Dirac operator D_Y on Y with real eigenvalues λ_i. The APS function

$$\eta(s) = \sum_i |\lambda_i|^{-s}\mathrm{sign}(\lambda_i), \tag{8.2}$$

where the sum runs over all nonzero λ_i, can be written

$$\eta(s) = \sum_{k\in\mathbf{Z}} \eta_k(s), \tag{8.3}$$

where $\eta_k(s)$ is the contribution from states that transform as $e^{-ik\theta}$ under rotation of the circle.

We write the spin bundle S of Y as $S = \pi^*(S_+) \oplus \pi^*(S_-)$, where S_+ and S_- are the positive and negative chirality spin bundles of X. Spinors on Y that transform as $e^{-ik\theta}$ under rotations of the circle are equivalent to spinors on X with values in $\mathcal{L}^k$. Let R be the radius of the $\mathbf{S}^1$, so the metric in the $\mathbf{S}^1$ direction is $R^2d\theta^2$. We can pick a basis of eleven-dimensional gamma matrices such that the Dirac operator reads

$$D_Y = \begin{pmatrix} \frac{i}{R}\frac{\partial}{\partial\theta} & \overline{D} \\ D & -\frac{i}{R}\frac{\partial}{\partial\theta} \end{pmatrix}, \tag{8.4}$$

where we have written the Dirac equation in 16×16 blocks, and we have arranged the spinors as a column vector

$$\begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}, \tag{8.5}$$

with $\psi_\pm$ being sections of $\pi^*(S_\pm)$. D and $\overline{D}$ are the ten-dimensional Dirac operators for positive and negative chirality. On spinors that transform as $e^{-ik\theta}$ under rotations of the circle, the Dirac equation $D_Y\psi = \lambda\psi$ becomes

$$\begin{pmatrix} \frac{k}{R} & \overline{D} \\ D & -\frac{k}{R} \end{pmatrix} \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} = \lambda \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}, \tag{8.6}$$

with $\psi_\pm$ being sections of $S_\pm \otimes \mathcal{L}^k$.

We recall that the phase of the M-theory action comes not just from η, but from $\eta + h$, where h is the number of zero eigenvalues. For $k = 0$, we have $\eta = 0$ for the same reason as in section 2. (To restate the argument in the present notation, the transformation $(\psi_+, \psi_-) \to (\psi_+, -\psi_-)$ maps $\lambda \to -\lambda$, so the nonzero eigenvalues occur in pairs.) The phase contribution for $k = 0$ therefore comes entirely from counting the zero eigenvalues. Since the spinors for $k = 0$ are sections of $S_\pm$, regardless of what $\mathcal{L}$ is, the contribution to the phase for $k = 0$ is independent of $\mathcal{L}$ and hence coincides with the phase of the effective action for a product $X \times \mathbf{S}^1$, as investigated in section 2.

For $k \neq 0$, instead, there are no zero eigenvalues, as is clear from inspection of (8.4), so the contributions will come entirely from the η invariant. The reason that it is possible to get a simple answer is that the nonzero eigenvalues of the ten-dimensional Dirac operator do not contribute even for $k \neq 0$. Suppose we have a pair of states $\psi_\pm$, which are sections of $S_\pm \otimes \mathcal{L}^k$, with $D\psi_+ = w\psi_-$, $D\psi_- = \overline{w}\psi_+$ for some complex number w. Then for these two states, the eleven-dimensional Dirac operator becomes

$$\begin{pmatrix} \frac{k}{R} & \overline{w} \\ w & -\frac{k}{R} \end{pmatrix}. \tag{8.7}$$

The $\eta(s)$ function of this 2×2 matrix is zero for any complex number w because the two eigenvalues have the same absolute value and opposite sign. So $\eta_k(s)$ for $k \neq 0$ can be computed entirely from the zero eigenvalues of the ten-dimensional Dirac operator.

Suppose now that ψ is a section of $S_+ \otimes \mathcal{L}^k$ or $S_- \otimes \mathcal{L}^k$ that is a zero mode of D or $\overline{D}$. We set $\chi(\psi)$ to be 1 or -1 depending on whether ψ has positive or negative chirality. ψ is an eigenstate of the eleven-dimensional Dirac operator with eigenvalue $k\chi(\psi)/R$. Its contribution to $\eta_k(s)$ is hence $|k/R|^{-s}\mathrm{sign}(k\chi) = |k/R|^{-s}\mathrm{sign}(k)\mathrm{sign}(\chi)$. When we sum

the quantity sign(χ) over all zero modes, we get the index of the ten-dimensional Dirac operator with values in $\mathcal{L}^k$; we denote this as $I(\mathcal{L}^k)$. So we have

$$\eta_k(s) = \left|\frac{k}{R}\right|^{-s} \mathrm{sign}(k) I(\mathcal{L}^k). \tag{8.8}$$

The function $\eta(s)$ is obtained by summing this expression over k. In doing so, we can observe that $I(\mathcal{L}^{-k}) = -I(\mathcal{L}^k)$. So we can express $\eta(s)$ as a sum over positive k only:

$$\frac{\eta(s)}{2} = \sum_{k=1}^{\infty} \left|\frac{k}{R}\right|^{-s} I(\mathcal{L}^k). \tag{8.9}$$

Now, the Atiyah-Singer index theorem gives a formula that in ten dimensions reads

$$I(\mathcal{L}^k) = \alpha k + \beta k^3 + \gamma k^5 \tag{8.10}$$

for certain rational numbers α, β, and γ. In particular, $I(\mathcal{L}^k)$ is a topological invariant. Together with the fact that the factor $|R|^s$ in (8.9) will play no role (as we will see shortly), this means that η will be a topological invariant.

Using (8.10), we have

$$\frac{\eta(s)}{2} = |R|^s \sum_{k=1}^{\infty} \left(\alpha k^{-(s-1)} + \beta k^{-(s-3)} + \gamma k^{-(s-5)}\right). \tag{8.11}$$

As expected, the series converges for sufficiently large Re(s). In fact, in terms of the Riemann zeta function ζ, we have

$$\frac{\eta(s)}{2} = |R|^s \left(\alpha\zeta(s-1) + \beta\zeta(s-3) + \gamma\zeta(s-5)\right). \tag{8.12}$$

This has the expected analytic continuation to $s = 0$. Since $\zeta(s)$ is regular at $s = -1, -3, -5$, the factor $|R|^s$ can be dropped. Using the values of $\zeta(-1)$, $\zeta(-3)$, and $\zeta(-5)$, we get

$$\frac{\eta}{2} = -\frac{\alpha}{12} + \frac{\beta}{120} - \frac{\gamma}{252}. \tag{8.13}$$

The above argument was presented for the Dirac operator, but it carries over in an obvious way to the Dirac operator coupled to any vector bundle V such that the bundle and connection are pulled back from X. Instead of $I(\mathcal{L}^k)$, we get $I(V \otimes \mathcal{L}^k)$ in the above formulas. If V is an E_8 bundle with characteristic class a, and if we set $e = c_1(\mathcal{L})$, then we have

$$I(V \otimes \mathcal{L}^k) = \int_X \left(248 + 60a + 6a^2 + \frac{1}{3}a^3\right) \hat{A}(X) e^{ke}. \tag{8.14}$$

(In dealing with rational or real cohomology classes, we will to keep the formulas short sometimes omit the cup or wedge product symbol.) Here, $\hat{A}(X)$ can be expanded

$$\hat{A}(X) = 1 + \hat{A}_4 + \hat{A}_8 = 1 - \frac{\lambda}{12} + \left(\frac{7\lambda^2 - p_2}{1440}\right). \tag{8.15}$$

We will find it convenient to express the formulas in terms of λ and $\hat{A}_8$.

The index formula (8.14) can be written as $\alpha k + \beta k^3 + \gamma k^5$ with

$$\begin{aligned} \alpha &= e(6a^2 + 60a\hat{A}_4 + 248\hat{A}_8) \\ \beta &= \frac{e^3}{6}\left(60a + 248\hat{A}_4\right) \\ \gamma &= 248\frac{e^5}{5!}. \end{aligned} \tag{8.16}$$

We also need the corresponding values for the Rarita-Schwinger operator. As explained in section 2, the Rarita-Schwinger operator on an eleven-manifold Y is, for our purposes, equivalent to the Dirac operator coupled to $TY - 3\mathcal{O}$, and for Y a circle bundle over X, it is equivalent to the Dirac operator coupled to $TX - 2\mathcal{O}$. (In string theory terms, $-2\mathcal{O}$ is the contribution of the ghosts plus the dilatino.) The appropriate index formula is therefore

$$I((TX - 2\mathcal{O}) \otimes \mathcal{L}^k) = \int_X \left(\sum_{i=1}^{5} 2\cosh(x_i) - 2\right) \hat{A}(X) e^{ke}, \tag{8.17}$$

where x_i are the Chern roots of TX, so $\lambda = p_1/2 = \sum_i x_i^2/2$ and $p_2 = \sum_{i<j} x_i^2 x_j^2$. We can evaluate the index formula as $\alpha' k + \beta' k^3 + \gamma' k^5$, with

$$\begin{aligned} \alpha' &= e(248\hat{A}_8 - \lambda^2) \\ \beta' &= \frac{2}{9}\lambda e^3 \\ \gamma' &= 8\frac{e^5}{5!}. \end{aligned} \tag{8.18}$$

These formulas can be used to evaluate the phase in (2.16).

8.2. *An Additional Phase*

The RR fields of Type IIA are expressed in terms of a K-theory class x by $G/2\pi = \sqrt{\hat{A}}\,\mathrm{ch}\,x$. In comparing to M-theory, we will assume that $G_0 = 0$ (since it has no known

M-theory interpretation), and hence to evaluate G_2 and G_4, we can set $\hat{A}$ to 1. We then get

$$\begin{aligned} \frac{G_0}{2\pi} &= 0 \\ \frac{G_2}{2\pi} &= c_1(x) \\ \frac{G_4}{2\pi} &= \frac{1}{2}c_1(x)^2 - c_2(x). \end{aligned} \tag{8.19}$$

In comparing M-theory to Type IIA, we will identify $c_1(x)$ with $e = c_1(\mathcal{L})$, and we will identify $-c_2(x)$ with the characteristic class a of the E_8 bundle over X. But $G_4/2\pi$ has an additional term $\frac{1}{2}c_1(x)^2$, and if we want to match M-theory with Type IIA, we need to include in the M-theory description an additional term that will shift $G/2\pi$ by $\frac{1}{2}c_1(\mathcal{L})^2$.

This additional term is topologically trivial, because in fact, $c_1(\mathcal{L})$, though nontrivial in the cohomology of X, pulls back to zero in the cohomology of the circle bundle $Y \to X$. Indeed, let ω be a one-form on Y that is $\mathbf{S}^1$ invariant and restricts on each fiber of $Y \to X$ to $d\theta/2\pi$. The normalization is picked so that

$$\int_{\mathbf{S}^1} \omega = 1, \tag{8.20}$$

where $\mathbf{S}^1$ is any fiber of $Y \to X$. Such an ω can be written as $\omega = (d\theta + A_i dx^i)/2\pi$, where x^i are coordinates on X and A is a connection on $\mathcal{L}$. We have $d\omega = F/2\pi$, where F is the curvature of $\mathcal{L}$; the fact that $F/2\pi = d\omega$ establishes (at the level of real cohomology) that $c_1(\mathcal{L})$ vanishes when pulled back to Y. (More generally, Y is the bundle of unit vectors in $\mathcal{L}$, and when pulled back to Y, $\mathcal{L}$ is trivialized tautologically.)

So if we set $C' = \pi\omega \wedge d\omega$, and $G' = dC'$, then $G'/2\pi = \frac{1}{2}F \wedge F/(2\pi)^2$. Adding C' to the C-field on Y has the effect, therefore, of shifting $G/2\pi$ by $\frac{1}{2}c_1(\mathcal{L})^2$. This is the shift we want.

Since C' is topologically trivial, the effect of the transformation $C \to C + C'$ on the phase of the M-theory effective action can be worked out from the form of the Chern-Simons coupling in a completely naive way. The Chern-Simons coupling is

$$L_{CS} = \frac{1}{6}\int_Y C \wedge \left(\left(\frac{G}{2\pi}\right)^2 - \frac{1}{8}\left(p_2(Y) - \lambda^2\right)\right). \tag{8.21}$$

If C is shifted by $C \to C + C'$ with C' topologically trivial, we can calculate directly that

$$\begin{aligned} L_{CS} \to & L_{CS} + \frac{1}{2}\int_Y C' \wedge \left(\left(\frac{G}{2\pi}\right)^2 - \frac{1}{24}\left(p_2(Y) - \lambda^2\right)\right) \\ & + \frac{1}{2}\int_Y C' \wedge \frac{dC'}{2\pi} \wedge \frac{G}{2\pi} + \frac{1}{6}\int_Y C' \wedge \frac{dC'}{2\pi} \wedge \frac{dC'}{2\pi}. \end{aligned} \tag{8.22}$$

Using $C' = \pi\omega \wedge d\omega$, together with (8.20) and the fact that $d\omega$ represents $e = c_1(\mathcal{L})$, we can evaluate the integral over the fibers of $Y \to X$ and find that the shift in L_{CS} due to C' is

$$\Delta L_{CS} = 2\pi \int_X \left\{ \frac{1}{4} e \left((a - \lambda/2)^2 - \frac{1}{24}(p_2 - \lambda^2) \right) + \frac{1}{8} e^3 (a - \lambda/2) + \frac{1}{48} e^5 \right\}. \tag{8.23}$$

Aggregate M-Theory Phase Factor

Combining the contributions of the η invariants, which give phase factors according to (2.16), with the phase we have just found in (8.23), the phase with which a configuration with specified $e = c_1(\mathcal{L})$ and characteristic class a of the M-theory four-form contributes to the partition function is

$$\Omega_M(e, a) = (-1)^{f(a)} \exp \left[2\pi i \int_X \left(\frac{e^5}{60} + \frac{e^3 a}{6} - \frac{11 e^3 \lambda}{144} - \frac{e a \lambda}{24} + \frac{e \lambda^2}{48} - \frac{e \hat{A}_8}{2} \right) \right]. \tag{8.24}$$

The exponential factor in (8.24) is unchanged if a is shifted by a torsion class. Therefore, the sum over torsion projects to a sum over $a = a_0 + b$ with $Sq^3 b = 0$, as before. We will compare the formidable-looking expression (8.24) to the Type IIA theta function in section 9.

8.3. Parity Symmetry

The discussion of parity symmetry in section 3.3 can be extended to $\mathbf{S}^1$ bundles Y over X as follows. Parity must now be interpreted as a reversal of orientation of the $\mathbf{S}^1$ fiber accompanied by $e \to -e$ and $G \to -G$. Combined with (8.19), this gives in terms of integral classes $a \to \lambda - e^2 - a$. Therefore, we have to check invariance of the phase (8.24) under $(e, a) \to (-e, \lambda - e^2 - a)$. Using the bilinear identity (3.13), we have

$$f(\lambda - e^2) = f(\lambda - e^2 - a) + f(a) + \int_X (\lambda - e^2 - a) \cup Sq^2 a. \tag{8.25}$$

The expression $\int_X e^2 \cup Sq^2 a$, with e, a integral classes, vanishes as a consequence of (4.26) and the Cartan formula (4.13), taking into account the fact that Sq^1 annihilates integral classes:

$$\int_X e^2 \cup Sq^2 a = \int_X Sq^2 e^2 \cup a = \int_X \big((Sq^2 e) \cup e + e \cup (Sq^2 e) \big) \cup a = 0. \tag{8.26}$$

Moreover, Stong's result (3.23) implies that

$$\int_X (\lambda - a) \cup Sq^2 a = 0. \tag{8.27}$$

Therefore, the last term in (8.25) vanishes. Repeating these steps for $f(\lambda - e^2)$, we find

$$f(\lambda) = f(\lambda - e^2 - a) + f(e^2) + f(a). \tag{8.28}$$

The variation of the additional phase factor in $\Omega_M(e, a)$, written in (8.24), can be evaluated by direct computation. Upon doing so and using (8.26), we find that $\Omega_M(e, a)$ transforms under parity by

$$\Omega_M(-e, \lambda - e^2 - a) = (-1)^{f(\lambda)+f(e^2)} \mathrm{exp}\left[2\pi i \int_X \left(\frac{2e^5}{15} - \frac{\lambda e^3}{18} + e\hat{A}_8\right)\right] \Omega_M(e, a). \tag{8.29}$$

The phase factor written as an exponential in (8.29) is in fact half the index density of the Dirac operator on X coupled to the K-theory class $\mathcal{L}^2 - \mathcal{O}$, where $\mathcal{O}$ denotes a trivial complex line bundle and $c_1(\mathcal{L}) = e$. Therefore we can rewrite (8.29) as

$$\Omega_M(-e, \lambda - e^2 - a) = (-1)^{f(\lambda)+f(e^2)+I\left(\mathcal{L}^2 - \mathcal{O}\right)} \Omega_M(e, a). \tag{8.30}$$

We will prove below, as part of a more general formula, that

$$f(e^2) = I\left(\mathcal{L}^2 - \mathcal{O}\right) \tag{8.31}$$

for all integral two-classes e. Therefore (8.30) reduces to

$$\Omega_M(-e, \lambda - e^2 - a) = (-1)^{f(\lambda)} \Omega_M(e, a), \tag{8.32}$$

which is the same as the result found in section 3.3 for trivial circle bundles. Thus inclusion of e does not modify the analysis of anomaly cancellation in section 3.3.

An Elementary Formula For The Mod Two Index Of A Product

The identity (8.31) needed above is part of a more general general formula expressing the mod two index of an E_8 bundle with characteristic class $a = u \cup v$ in terms of elementary invariants. Here u, v are integral two-classes in $H^2(X; \mathbf{Z})$.

Such a formula can be derived by constructing E_8 bundles using the embedding $SU(3) \subset E_8$, in analogy with the proof of the bilinear identity in section 3.1. Let $\mathcal{L}$,

$\mathcal{M}$ be complex line bundles with $c_1(\mathcal{L}) = u$, $c_1(\mathcal{M}) = v$. We first construct the $SU(3)$ bundle

$$W = \mathcal{L} \oplus \mathcal{M} \oplus \overline{\mathcal{L}} \otimes \overline{\mathcal{M}}. \tag{8.33}$$

A direct computation shows that

$$c_2(W) = -(u^2 + v^2 + u \cup v). \tag{8.34}$$

Therefore, by embedding $SU(3)$ in E_8 (using the chain $SU(3) \subset E_6 \times SU(3) \subset E_8$) we obtain an E_8 bundle with characteristic class $a = u^2 + v^2 + u \cup v$.

The decomposition of the Lie algebra of E_8 in terms of representations of $SU(3) \times E_6$ is

$$\mathbf{248} = (\mathbf{8}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{78}) \oplus (\mathbf{3}, \overline{\mathbf{27}}) \oplus (\overline{\mathbf{3}}, \mathbf{27}). \tag{8.35}$$

The mod two index of the E_8 bundle constructed above is the same as the mod two index with values in the $\mathbf{8} \oplus \mathbf{3} \oplus \overline{\mathbf{3}}$ of $SU(3)$. This can be evaluated using the fact that the mod 2 index with values in $\mathcal{S} \oplus \overline{\mathcal{S}}$ (for any $\mathcal{S}$) is the mod 2 reduction of the ordinary index with values in $\mathcal{S}$. We get

$$\begin{aligned} f(u^2 + v^2 + u \cup v) =& I\left(\mathcal{L}^2 \otimes \mathcal{M} \oplus \mathcal{L} \otimes \mathcal{M}^2\right) + I\left(\mathcal{L} \otimes \overline{\mathcal{M}} \oplus \mathcal{L} \otimes \mathcal{M}\right) \\ &+ I\left(\mathcal{L} \oplus \mathcal{M}\right) \qquad \text{mod } 2. \end{aligned} \tag{8.36}$$

As an ordinary index, the right hand side of equation (8.36) can be expressed in terms of elementary invariants. Setting $\mathcal{M} = \mathcal{O}$, and working mod two, we obtain

$$f(u^2) = I\left(\mathcal{L}^2 - \mathcal{O}\right) \qquad \text{mod } 2. \tag{8.37}$$

This is the formula (8.31) needed above.

We record here a more general identity which is easily obtained from (8.36) and (8.37) using the bilinear identity for f and the index theorem. Applying twice the bilinear identity, and taking into account (8.26), we have

$$f(u^2 + v^2 + u \cup v) = f(u^2) + f(v^2) + f(u \cup v). \tag{8.38}$$

Combining (8.36) – (8.38), we arrive at

$$\begin{aligned} f(u \cup v) =& I\left((\mathcal{L} - \mathcal{O}) \otimes (\mathcal{M}^2 - \mathcal{O})\right) + I\left((\mathcal{L}^2 - \mathcal{O}) \otimes (\mathcal{M} - \mathcal{O})\right) \\ &+ I\left(\mathcal{L} \otimes \overline{\mathcal{M}} \oplus \mathcal{L} \otimes \mathcal{M}\right) \qquad \text{mod } 2. \end{aligned} \tag{8.39}$$

The right hand side of (8.39) can be evaluated using the index theorem, obtaining

$$f(uv) = \int \left[uv(u+v)\left(uv - \frac{1}{4}\lambda\right) + \frac{3}{4}uv(u^3+v^3) + \frac{1}{12}\left(uv^4 + 2u^3v^2 - \lambda uv^2\right)\right] \quad \text{mod2} \tag{8.40}$$

The right hand side of this formula is symmetric under exchanging u and v since on a spin 10-manifold $I((\mathcal{L} \otimes \overline{\mathcal{M}}) = I\left(\overline{\mathcal{L}} \otimes \mathcal{M}\right)$ mod 2.

9. Generalized Comparison To Type IIA

In this section, we show how the computation of section 8 is reproduced in the IIA theory. Our aim is to obtain the nontrivial phase (8.24) using the K-theory formalism. The real part of the action will match between M-theory and IIA theory simply because the dimensional reduction of 11 dimensional supergravity is the IIA supergravity.

The relation between the M-theory geometry and the RR fields has already been explained in section 8.2. A K-theory class x satisfying (8.19) corresponds to M-theory on a circle bundle $Y \to X$, where the Euler class of the circle bundle is $c_1(x)$. Moreover, G_4 is pulled back to Y to determine the M-theoretic G. We therefore must compute the contribution (7.24) for such x, and compare to (8.24).

At $G_2 = 0$, we have found it necessary to compare an E_8 bundle associated with M-theory to an $SU(5)$ bundle derived from K-theory. As in section 7, we write the characteristic class a of the E_8 bundle as $a = a_0 + b$, where a_0 was defined in section 6 and b has a K-theory lift. By the relation (8.19), we see that we must choose our K-theory class to be represented by

$$x = E + \mathcal{L} - 6\mathcal{O} + \Delta \tag{9.1}$$

Here E is the $SU(5)$ bundle used in section 7.5, with $c_2(E) = -b$. Also, $\mathcal{L}$ is a line bundle on X with $c_1(\mathcal{L}) = e$; a rank six trivial bundle $6\mathcal{O}$ has been subtracted to ensure $G_0 = 0$. Finally, Δ is a class in Γ_2 chosen so that $x \in \Gamma_1$. One can check, as in section 7, that the choice of Γ_1, i.e. the choice of Δ, will not contribute to the phase Z_x, so we can ignore Δ. With this understanding, we can write

$$x = x_0 + (\mathcal{L} - \mathcal{O}) \tag{9.2}$$

where $x_0 = E - 5\mathcal{O}$ is the class used in section 7.

The evaluation of the contribution Z_x to the partition function requires evaluation of $\Omega(x)$ and Re $(\tau(x))$. Let us consider first $\Omega(x)$. Using the bilinear identity we have

$$\Omega(x) = \Omega(x_0)\Omega(\mathcal{L} - \mathcal{O})e^{-i\pi I(x_0 \otimes (\overline{\mathcal{L}} - \mathcal{O}))}. \tag{9.3}$$

Now, $(\mathcal{L} - \mathcal{O}) \otimes (\overline{\mathcal{L}} - \mathcal{O}) = 2 - (\mathcal{L} + \overline{\mathcal{L}})$. Therefore $\Omega(\mathcal{L} - \mathcal{O})$ is elementary; it equals the reduction modulo two of the ordinary index $I(\mathcal{L})$. So we have

$$\Omega(x) = \Omega(x_0)e^{-i\pi [I(x_0 \otimes (\overline{\mathcal{L}} - \mathcal{O})) + I(\mathcal{L})]}. \tag{9.4}$$

Now we can use the index theorem. Substituting

$$\mathrm{ch}x_0 = b + \frac{1}{2}c_3(E) + \left(\frac{b^2 - 2c_4(E)}{12}\right) + \frac{c_5(E)}{24} \tag{9.5}$$

and applying the result (7.33) to evaluate $\Omega(x_0)$, we conclude

$$\begin{aligned}\Omega(x) = \exp\Bigg[i\pi \Bigg(& f(b) + \frac{1}{2}(\lambda - b)c_3 \\ & -[\frac{1}{6}c_4 e + \frac{1}{4}c_3 e^2 - \frac{1}{6}be^3 - \frac{1}{12}eb^2 + \frac{1}{12}eb\lambda] \\ & -[\frac{e^5}{5!} - \frac{\lambda e^3}{72} + e\hat{A}_8] \Bigg) \Bigg]\end{aligned} \tag{9.6}$$

Here $c_4 = c_4(E)$, $c_5 = c_5(E)$.

Let us now turn to the contribution of Re $(\tau(x))$. This is a straightforward application of the general formula (7.20):

$$\mathrm{Re}\tau\left(x + \frac{1}{2}\theta\right) = \frac{1}{(2\pi)^2}\int (G_4 G_6 - G_2 G_8)\,. \tag{9.7}$$

In terms of Chern classes we have

$$\begin{aligned}\frac{G_2}{2\pi} &= \mathrm{ch}_1\left(x + \frac{1}{2}\theta\right) \\ \frac{G_4}{2\pi} &= \mathrm{ch}_2\left(x + \frac{1}{2}\theta\right) \\ \frac{G_6}{2\pi} &= \mathrm{ch}_3\left(x + \frac{1}{2}\theta\right) - \frac{\lambda}{24}\mathrm{ch}_1\left(x + \frac{1}{2}\theta\right) \\ \frac{G_8}{2\pi} &= \mathrm{ch}_4\left(x + \frac{1}{2}\theta\right) - \frac{\lambda}{24}\mathrm{ch}_2\left(x + \frac{1}{2}\theta\right)\end{aligned} \tag{9.8}$$

The contributions (proportional to λ) from $\hat{A}$ cancel, and hence we get

$$\begin{aligned}\mathrm{Re}\tau\left(x + \frac{1}{2}\theta\right) = \int & (\mathrm{ch}_2(x) + \frac{1}{2}\mathrm{ch}_2(\theta))(\mathrm{ch}_3(x) + \frac{1}{2}\mathrm{ch}_3(\theta)) \\ & - (\mathrm{ch}_1(x) + \frac{1}{2}\mathrm{ch}_1(\theta))(\mathrm{ch}_4(x) + \frac{1}{2}\mathrm{ch}_4(\theta)).\end{aligned} \tag{9.9}$$

Now we expand the expression (9.9). We get three kinds of terms. The quadratic piece in the Chern classes of x is

$$\int (\mathrm{ch}_2(x)\mathrm{ch}_3(x) - \mathrm{ch}_1(x)\mathrm{ch}_4(x))\,. \tag{9.10}$$

We can simplify the cross terms in (9.9) using the orthogonality relations since θ and x are both in the Lagrangian lattice Γ_1. We use orthogonality to eliminate $\mathrm{ch}_3(\theta)$ and $\mathrm{ch}_4(\theta)$ and get the cross terms:

$$\int \left(\mathrm{ch}_2(\theta)\mathrm{ch}_3(x) - \mathrm{ch}_1(\theta)\mathrm{ch}_4(x) - \frac{\lambda}{24}\big(\mathrm{ch}_1(x)\mathrm{ch}_2(\theta) - \mathrm{ch}_1(\theta)\mathrm{ch}_2(x)\big) \right). \tag{9.11}$$

Finally there are the terms quadratic in θ:

$$\frac{1}{4}\left(\mathrm{ch}_2(\theta)\mathrm{ch}_3(\theta) - \mathrm{ch}_1(\theta)\mathrm{ch}_4(\theta)\right). \tag{9.12}$$

Now we write out these expressions in terms of Chern *classes* (as opposed to characters). The quadratic piece in the Chern classes of x is

$$\mathrm{ch}_2(x)\mathrm{ch}_3(x) - \mathrm{ch}_1(x)\mathrm{ch}_4(x) = \frac{1}{6}c_4 e + \frac{1}{4}c_3 e^2 + \frac{1}{2}c_3 b - \frac{1}{12}eb^2 + \frac{1}{6}be^3 + \frac{1}{24}e^5 \tag{9.13}$$

Here c_i is an abbreviation for $c_i(x_0) = c_i(E)$ (not the Chern classes of x).

We now simplify the cross terms using $\mathrm{ch}_2(\theta) = -\lambda + 2a_0$ to get

$$-\lambda\left(\frac{1}{2}c_3 + \frac{1}{6}e^3\right) + \frac{\lambda^2}{24}e + a_0\left(c_3 + \frac{e^3}{3} - \frac{\lambda e}{12}\right). \tag{9.14}$$

The piece quadratic in the Chern classes of θ cancels the first factor in (7.24).

Combining (9.6), (9.13) and (9.14) and using $b = a - a_0$ we find

$$Z_x = \exp\left[i\pi\left(f(b) + \int a_0 c_3\right)\right] \cdot \exp\left[2\pi i \int_X \left(\frac{e^5}{60} + \frac{e^3 a}{6} - \frac{11 e^3 \lambda}{144} - \frac{ea\lambda}{24} + \frac{e\lambda^2}{48} - \frac{e\hat{A}_8}{2}\right)\right] \tag{9.15}$$

Finally, we use the bilinear identity to conclude that

$$(-1)^{f(a)} = (-1)^{f(a_0)} e^{i\pi(f(b) + \int a_0 c_3)} \tag{9.16}$$

The sign $(-1)^{f(a_0)}$ is part of the overall manifold-dependent normalization which we are not trying to match (see, e.g. (7.44)). Apart from this, comparison of (9.15) with (8.24) yields a perfect match. This completes the comparison of the K-theory and M-theory phase factors for nonzero values of G_2.

10. Completing the Type IIA Theta Function

In this section, we will extend the computation of sections 7 and 9 above to include the effects of nonzero G_0, thus completing the formula for the full Type IIA theta function. On the one hand, the effects of G_0 are the least important in the large volume limit, being of order $\exp[-G_0^2 V]$ where V is the volume of X in string units. On the other hand, while most of this paper has focused on the interesting subtleties related to $H^4(X;\mathbf{Z})$, it is worth noting that the contribution from G_0 is the only nontrivial contribution to the theta function for such basic manifolds as $X = \mathbf{S}^{10}$ and $X = \mathbf{S}^5 \times \mathbf{S}^5$. Moreover, as we leave the geometrical realm and make the volume smaller, these are the *most* important terms. We comment on the relation to M-theory at the end of this section.

First, let us construct the full maximal Lagrangian lattice $\Gamma_1 \subset K(X)$. As we have seen, for all $c_1 \in H^2(X;\mathbf{Z})$ and $c_2 \in H^4(X;\mathbf{Z})$ with $Sq^3 c_2 = 0$, there is a K-theory lift in Γ_1, that is, there is a K-theory class $x(c_1, c_2) \in \Gamma_1$ with

$$\mathrm{ch}(x(c_1, c_2)) = c_1 + (-c_2 + \frac{1}{2}c_1^2) + \cdots \tag{10.1}$$

where the higher Chern classes are such that x is in a Lagrangian lattice. Now, we may choose the K-theory lifts $x(c_1, c_2)$ such that, for all c_1, c_2, the index $I(x)$ of the Dirac operator with values in x is zero. This is possible because on any ten-manifold X, there exists a K-theory class y, trivial except in a small neighborhood of a point in X, with index 1. The Chern classes of y vanish except for $\frac{1}{4!}c_5(y) = 1$. By adding to x a multiple of y, one can pick the K-theory lifts so that $I(x) = 0$ for all x. Similarly we can take $I(\theta) = 0$. Once this is done, one can define a complete Lagrangian lattice Γ_1 that consists of K-theory classses of the form $z = m\mathcal{O} + x(c_1, c_2)$ where $m \in \mathbf{Z}$ and $\mathcal{O}$ is a trivial complex line bundle.

We have already computed the contribution of $x(c_1, c_2)$ to the partition function in sections 7 and 9. Let us see what changes by including $m\mathcal{O}$. Now we have:

$$\frac{G(z + \frac{1}{2}\theta)}{2\pi} = m\sqrt{\hat{A}} + \frac{G(x + \frac{1}{2}\theta)}{2\pi} \tag{10.2}$$

Moreover, using

$$0 = I(x) = \int \frac{G(x)}{2\pi}\sqrt{\hat{A}} \tag{10.3}$$

we get

$$\left(G_0G_{10} - G_2G_8 + G_4G_6\right)\Big|_{z+\frac{1}{2}\theta} = \left(G_4G_6 - G_2G_8\right)\Big|_{x+\frac{1}{2}\theta} \\ - 4\pi m\big(\sqrt{\hat{A}}\big)_8 G_2(x) \tag{10.4}$$

Similarly, by the cocycle formula and the fact that $\omega(\mathcal{O}, x) = I(x) = 0$, we have

$$\Omega_{IIA}(m\mathcal{O} + x) = (\Omega_{IIA}(\mathcal{O}))^m \Omega_{IIA}(x). \tag{10.5}$$

Note that, if $\Omega_{IIA}(\mathcal{O}) = -1$, then the dilatino has a zero mode, and the partition function vanishes. Even when this occurs, the total number of fermion zero modes is still even (because Type IIA has fermions coming from both left- and right-movers on the world-sheet), and by insertion of a local operator, we can obtain nonzero and sensible correlation functions.

The most interesting change from the previous sections is in the kinetic energy, which now reads:

$$\frac{G}{2\pi} \cdot \mathrm{Im}\tau \cdot \frac{G}{2\pi} = m^2 V + \left|\frac{G_2(x)}{2\pi}\right|^2 + \left|\frac{G_4(x + \frac{1}{2}\theta)}{2\pi} - m\frac{\lambda}{24}\right|^2 \tag{10.6}$$

where V is the volume of X_{10} in the string metric in string units.

Thus, assuming $\Omega_{IIA} = 1$ on $K^0(X)_{tors}$, the full IIA theta function becomes

$$\Theta_{IIA} = \sum_{c_1, c_2 : Sq^3 c_2 = 0} w(c_1, c_2) e^{-i\pi\tau\theta(c_1,c_2)^2 - 2\pi i\theta(c_1,c_2)\varphi(c_1)} \vartheta \begin{bmatrix} \theta(c_1, c_2) \\ \varphi(c_1) \end{bmatrix} (0|\tau) \tag{10.7}$$

where $w(c_1, c_2)$ is the weighting factor computed previously (see equations (7.20) and (9.15) above). The effect of the sum over G_0 is to change the weighting factor to an *elliptic function*, namely a theta function with

$$\tau = i\left(V + |\frac{\lambda}{24}|^2\right) \tag{10.8}$$

and characteristics

$$\theta(c_1, c_2) = -\frac{1}{2\pi}\frac{\int G_4(x + \frac{1}{2}\theta) \wedge *\frac{\lambda}{24}}{V + |\frac{\lambda}{24}|^2} \\ \varphi(c_1) = -\int (\sqrt{\hat{A}})_8 c_1 + \varphi_0 \tag{10.9}$$

where $\Omega(\mathcal{O}) = \exp[2\pi i\varphi_0]$.

10.1. No Comparison to M-Theory

Unlike the results of sections 7 and 9, we cannot, unfortunately, make a comparison with M-theory. The reason is that the sectors with nonzero G_0 correspond to sectors of Type IIA supergravity with nonzero Romans mass [32,33]. There is no accepted M-theoretic background corresponding to such IIA backgrounds. Nevertheless, the form of the above answer is somewhat suggestive, so we offer one speculation.

The appearance of elliptic functions is suggestive of F-theory and M-theory geometries involving torus bundles. Indeed, for certain manifolds, η invariants are closely related to L-functions and modular functions [34]. Essentially, this arises because $\eta(s)$ is a generalized Dirichlet series and can be evaluated via a generalization of the Kronecker limit formula.

A relation between M-theory on torus bundles and massive IIA string theory has in fact been suggested by Hull (in some special backgrounds) [35]. One way of interpreting Hull's result is in terms of T-duality which can relate IIA theory on $T^2 \times X_8$ with $G_0/2\pi = m$, $G_2 = 0$ to IIA theory on a dual torus with $G_0 = 0$ and $G_2/(2\pi) = me_0$, where e_0 generates $H^2(T^2; \mathbf{Z})$. [18] The latter geometry (for a large dual torus) has an M-theoretic interpretation which we have analyzed. Using the above results one can check that the actions of T-dual geometries do not agree, but appropriate sums of such actions do agree. In this sense we can confirm the suggestion of [35].

11. Some Remarks About The H-Field And A Puzzle

Though the present paper primarily focuses on the case that the Neveu-Schwarz three-form field H (and in fact the NS potential B) is zero, we will here make a few simple observations about what happens when it is included.

First of all, in supergravity, as explained in [36], the equations of motion for the RR fields G_n can be put in the form

$$dG_{n+2} = H \wedge G_n \tag{11.1}$$

for all n. At the level of cohomology, this implies simply that

$$H \wedge G_n = 0. \tag{11.2}$$

[18] It is an interesting and not entirely trivial exercise to demonstrate explicitly the $SO(2,2;\mathbf{Z})$ T-duality invariance of the K-theory theta function on manifolds of the form $T^2 \times X_8$.

Let us compare this to what we might expect for D-branes. We simply repeat the reasoning of section 5.1, but now with $H \neq 0$. In the presence of an H-field, D-brane charge takes values in a twisted K-group K_H (which can be defined straightforwardly [27] when H is torsion and less straightforwardly [37,38] when it is not). What D-branes represent classes in K_H? In the presence of an H-field, a D-brane can be wrapped on a cycle Q if and only if [22]

$$H|_Q + W_3(N) = 0. \tag{11.3}$$

Here $H|_Q$ is the restriction of H to Q, and N is the normal bundle to Q. In terms of a cohomology class b that is Poincaré dual to Q, this equation amounts to

$$(H + Sq^3)b = 0. \tag{11.4}$$

This is the condition, in the presence of the H-field, for b to represent the lowest nonvanishing p-form charge of a D-brane. Of course, b is of odd or even degree for Type IIA or Type IIB and so represents an element of K^1_H or K_H, respectively. We interpret the operator $H+Sq^3$ that appears in (11.4) as the first differential d_3 of the Atiyah-Hirzebruch spectral sequence for K_H; it reduces at $H = 0$ to the familiar Sq^3.

If RR charges have a K-theory interpretation, then the fields that the charges create must also have a K-theory interpretation. Using arguments along the lines of those in section 2 of [2], we may expect the RR forms G_n to themselves be elements of K_H – more exactly, elements of K_H or K^1_H for Type IIA or Type IIB. Since we have identified the first AHSS differential as $H + Sq^3$, it follows that in any physical situation in which the G_p vanish for $p < n$, we should have at the level of cohomology

$$(H + Sq^3)G_n = 0. \tag{11.5}$$

We interpret this as the supergravity equation (11.2) with a torsion correction Sq^3. (If $G_n \neq 0$, then for $m > n$, the conditions on G_m are more complicated in general and will involve higher order effects in the AHSS.)

Clearly, it would be desireable to generalize the computation in sections 7 and 9 and show that the M-theory partition function on a circle bundle over X, with $H \neq 0$, can be expressed in terms of RR fields obeying (11.5). This will not be demonstrated in the present paper, although we have performed several computations which do in fact support this hypothesis. In particular, for $Y = X \times \mathbf{S}^1$ and H 2-torsion we have generalized the computation of section 7.

A Puzzle

We will now point out a puzzle that this picture raises involving S-duality or $SL(2,\mathbf{Z})$ symmetry for Type IIB. First of all, to have $SL(2,\mathbf{Z})$ symmetry, we must assume that the monodromies of the Type IIB τ parameter are trivial. This forces us to assume that the cohomology class G_1 (which determines the monodromies of $\mathrm{Re}\,\tau$) vanishes. This being so, the lowest RR form that may be topologically non-trivial is G_3, which we will call simply G. The equation (11.5) hence implies that at the level of cohomology we should have

$$(H + Sq^3)G = 0. \tag{11.6}$$

Now, at least modulo torsion, the pair $\begin{pmatrix} G \\ H \end{pmatrix}$ is expected to transform in the two-dimensional representation of $SL(2,\mathbf{Z})$. The transformation $G \to G + H$, $H \to H$ corresponds to $\tau \to \tau + 1$ and is visible in string perturbation theory. Happily, (11.6) is invariant under this transformation (since $H \cup H = Sq^3 H$). The problem arises because (11.6) does *not* have full $SL(2,\mathbf{Z})$ symmetry; it is not invariant under $G \to G$, $H \to H + G$. (11.6) has an $SL(2,\mathbf{Z})$-invariant extension, namely

$$H \cup G + Sq^3(G + H) = 0. \tag{11.7}$$

(More generally, one should allow for a G,H-independent constant on the right hand side of (11.7) analogous to P introduced in section 6.2.) Unfortunately, it is hard to see a rationale for the $Sq^3 H$ term in this equation.

The root of the problem is that the weak coupling description in which the RR fields are classified by K^1_H breaks the symmetry between G and H by treating H as an "ordinary three-form field," while G is a more subtle object related to K-theory.

We do not know where the resolution of this problem may lie. We can see at least two broad approaches to resolving the problem:

(1) Perhaps (11.7) is correct. In support of this hypothesis, we note that the sort of arguments given in [2] only show that RR fields can be classified by $K_H(X)$ (or by $K(X)$ if $H = 0$) modulo an additive constant. Roughly, the arguments in section 2 of [2] show that the RR fields created by a D-brane are classified by K_H, but there may be a "background field" or integration constant, not created by the D-branes and not obeying (11.5). Thus, (11.5) would be replaced by $(H + Sq^3)G_n = Q$, for some class Q that should be independent of G_n. (In sections 7 and 9, we found a superficially similar shift by P, which was interpreted in terms of the "characteristic" of the theta function; it does not

seem that the $H \cup H$ term has such an interpretation.) If, for Type IIB with $n = 3$, one has $Q = H \cup H$, then we would arrive at (11.7). Moreover, in the special case of M-theory on backgrounds of the form $\mathbf{T}^2 \times X_9$ and H 2-torsion we have in fact derived (11.7) from the M-theory phase. Unfortunately, in the general case we have not been able to turn this idea into a coherent proposal, or to find convincing support for it.

(2) Alternatively, perhaps $SL(2, \mathbf{Z})$ invariance of the theory does not come from a simple transformation law on the space of classical fields. On a compact ten-manifold X, the partition function in the large volume limit is hopefully $SL(2, \mathbf{Z})$-invariant. (We take large volume on X to reduce to a situation in which supergravity, perhaps with some corrections such as Sq^3, should be valid, and equations such as (11.5) make sense.) This invariance need not come from an $SL(2, \mathbf{Z})$ action on the classical fields. For example, in comparing M-theory to Type IIA, we did not find a simple matching between M-theory configurations and Type IIA configurations; we have had to identify the contribution of an equivalence class of M-theory fields (described in section 6) with the contribution of an equivalence class of Type IIA configurations (classified by an element of $K(X)/K(X)_{tors}$). However, we have not been able to find a convincing scenario for $SL(2, \mathbf{Z})$ symmetry of the partition function (or of the Hilbert space in a Hamiltonian description) without an $SL(2, \mathbf{Z})$ action on the configuration space.

12. Conclusions and Questions

In summary, let us recapitulate some of the main lessons we have learned from the above considerations, and raise some questions.

One key point is that extremely subtle phases that are not generated by any conventional supergravity Lagrangian are essential in a careful comparison of Type IIA superstrings and M-theory. There are descriptions of these phases via gauge theory – $U(N)$ gauge theory and E_8 gauge theory for Type IIA and M-theory, respectively. By reconciling the Type IIA and M-theory formulas, we have gained considerable confidence that both are correct.

We have developed a more complete understanding of the conditions on allowed RR fluxes coming from the K-theory interpretation of RR fields. As a byproduct, we have learned that certain apparently stable D-brane configurations are actually unstable.

Is there a physical mechanism that would naturally generate the phases required in Type IIA and in M-theory? Is the use of E_8 gauge fields to describe M-theory C-fields

merely a technical device, or is there an underlying physical meaning to this? (See [39] for some speculations related to this question.) Likewise, what is the deeper meaning of the $U(N)$ gauge fields that are implicit in the K-theory description of RR fields?

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Appendix A. Notation

In this paper, cohomology classes with integer coefficients or unspecified coefficients will generally be labeled as a, b, c. The symbols $\overline{a}, \overline{b}, \overline{c}$ will denote cohomology classes with $\mathbf{Z}_2$ coefficients. K-theory classes will be denoted as x, y, z, and complex vector bundles as E, F. The K-theory class determined by a pair of bundles E and F will be written as (E, F) or $E - F$.

A list of selected notation used throughout the paper is:

$a, a', b, b', \ldots$	Generic elements of $H^4(X, Z)$.
A	$Sq^3(H^4(X, Z))$. Also, a gauge field.
B	$Sq^3(H^4(X, Z)_{tors})$
c	A generic torsion cohomology class
C	The M-theory 3-field potential.
e	The first Chern class of a circle bundle (secs. 8 and 9)
G	The M-theory 4-form field-strength. Also the total IIA RR field-strength.
G_{2p}	The $2p$-form RR field-strength in IIA
Γ	$K(X)/K(X)_{tors}$

$I(\mathcal{E};X)$ Index of an elliptic operator $\mathcal{E}$ on X

$I(x), I(x;X)$ Index of the Dirac operator coupled to $x \in K(X)$.

$j(x)$ Mod two index of Dirac coupled to $x \otimes \overline{x}$.

L $H^4_{tors}/2H^4_{tors}$

M 2-torsion subgroup of $H^7_{tors}(X,Z)$.

$\mathcal{O}$ A trivial line bundle (real or complex depending on the context).

P A cohomology class in $H^7(X,Z)/Sq^3(H^4_{tors})$ defined in (6.7).

$q(x;X), q(x)$ The mod-two Dirac index with values in $x \in KO(X)$.

S $H^4(X,Z)/H^4_{tors}(X,Z)$

S' $\{a \in H^4(X,Z)/H^4_{tors}(X,Z) : Sq^3 a \in Sq^3 H^4(X,Z)_{tors}\}$

T H^6/H^6_{tors}, the torsion pairing (or in section 5.2 the tachyon field).

Υ_0 The kernel of Sq^3 on $H^4(X,Z)_{tors}$.

Υ The subspace of Υ_0 such that $Sq^2 b$ has a *torsion* integral lift.

$V(a)$ An E_8 vector bundle on X determined by $a \in H^4(X,\mathbf{Z})$.

x, y, z Generic elements of $K(X)$.

X, Y, Z A spin manifold, usually of dimension 10, 11, or 12 respectively.

Appendix B. Computation Of Some Mod 2 Indices

To complete the proof begun in section 3.3 that the symmetry $(-1)^{F_L}$ of Type IIA superstring theory is anomaly free, we must prove that the anomaly vanishes for a manifold $V_{1,1}$ defined as a hypersurface of degree $(1,1)$ in $\mathbf{CP}^2 \times \mathbf{CP}^4$. We let x_i, $i = 1, \dots, 3$ be homogeneous coordinates for $\mathbf{CP}^2$, and we let y_j, $j = 1, \dots, 5$, be homogeneous coordinates for $\mathbf{CP}^4$. We can take the equation defining $V_{1,1}$ to be

$$\sum_{i=1}^{3} x_i y_i = 0. \tag{B.1}$$

For every point $(x_1, x_2, x_3) \in \mathbf{CP}^2$, the x_i are not all zero, and this equation defines a hyperplane in $\mathbf{CP}^4$, which is isomorphic to $\mathbf{CP}^3$. Hence, $V_{1,1}$ can be viewed as a $\mathbf{CP}^3$ bundle over $\mathbf{CP}^2$.

We can pick a metric on $V_{1,1}$ in which the fibers of $V_{1,1} \to \mathbf{CP}^2$ are small and have positive scalar curvature. In such a metric, the Dirac operator on $V_{1,1}$ has no zero modes, and hence the mod 2 index $q(\mathcal{O})$ vanishes. We wish to show that the mod 2 indices $q(T)$ and $q(\wedge^2 T)$ also vanish.

For this we use the fact that $V_{1,1}$ is a complex manifold, and the complexification of T splits as $T \otimes_{\mathbf{R}} \mathbf{C} = \mathcal{T} \oplus \overline{\mathcal{T}}$, where $\mathcal{T}$ is the holomorphic tangent bundle to $V_{1,1}$. Hence, $q(T)$ is the mod 2 reduction of the ordinary index $I_{\mathcal{T}}$ of the Dirac operator on $V_{1,1}$ with values in $\mathcal{T}$. Likewise $\wedge^2 T \otimes_R \mathbf{C} = \wedge^2 \mathcal{T} \oplus \wedge^2 \overline{\mathcal{T}}$, and hence $q(\wedge^2 T)$ is the mod 2 reduction of the ordinary index $I_{\wedge^2 \mathcal{T}}$ of the Dirac operator on $V_{1,1}$ with values in $\wedge^2 \mathcal{T}$. We will show that $I_{\mathcal{T}}$ vanishes, and that $I_{\wedge^2 \mathcal{T}}$ is even. The strategy will be to compute the index by first solving the Dirac equation along the fibers of $V_{1,1} \to \mathbf{CP}^2$ to get an "index bundle" $W \to \mathbf{CP}^2$; then we compute the index of the Dirac operator on $V_{1,1}$ as the index of the Dirac operator on $\mathbf{CP}^2$ with values in W. A special case of this procedure is that the Dirac index on $V_{1,1}$ with values in any bundle F that is trivial when restricted to each fiber of $V_{1,1} \to \mathbf{CP}^2$ vanishes. This can be proved rather as above by picking on $V_{1,1}$ a metric such that the fibers are small and have positive scalar curvature; in this metric, the Dirac operator on $V_{1,1}$ with values in F has no zero modes at all, so the index bundle is trivial.

We will first show that $I_{\mathcal{T}} = 0$. Because of the fibration of $V_{1,1}$ over $\mathbf{CP}^2$ with $\mathbf{CP}^3$ fibers, there is an exact sequence

$$0 \to \mathcal{T}\mathbf{CP}^3 \to \mathcal{T}V_{1,1} \to \mathcal{T}\mathbf{CP}^2 \to 0. \tag{B.2}$$

This sequence does not split holomorphically, but it does split topologically and as we are just doing index theory, we can replace $\mathcal{T}V_{1,1}$ by $\mathcal{T}\mathbf{CP}^3 \oplus \mathcal{T}\mathbf{CP}^2$. Now, $\mathcal{T}\mathbf{CP}^2$ is a pullback from the base of $V_{1,1} \to \mathbf{CP}^2$, so it is trivial on each fiber of $V_{1,1}$ and hence by a remark above has zero index. We still have to look at the index with values in $\mathcal{T}\mathbf{CP}^3$. In fact, taking a metric on $\mathbf{CP}^3$ that is Kähler, the Dirac operator on $\mathbf{CP}^3$ with values in $\mathcal{T}\mathbf{CP}^3$ has no zero modes. This operator is indeed equivalent to the $\overline{\partial}$ operator acting on $\mathcal{T}\mathbf{CP}^3 \otimes K^{1/2}$, where K is the canonical bundle of $\mathbf{CP}^3$. As $K \cong \mathcal{O}(-4)$, we must show that $H^i(\mathbf{CP}^3, \mathcal{T}\mathbf{CP}^3(-2)) = 0$ for all i. For this, we use the existence of an exact sequence

$$0 \to \mathcal{O} \to \mathcal{O}(1)^4 \to \mathcal{T}\mathbf{CP}^3 \to 0. \tag{B.3}$$

This exact sequence expresses the fact that a tangent vector field on $\mathbf{CP}^3$ can be written as

$$\sum_{i=1}^{4} a_i \frac{\partial}{\partial y_i}, \tag{B.4}$$

where the a_i are functions of the y's that are homogeneous of degree 1, and defined up to $a_i \to a_i + wy_i$ for any complex function w. A twisted version of this exact sequence reads

$$0 \to \mathcal{O}(-2) \to \mathcal{O}(-1)^4 \to \mathcal{T}\mathbf{CP}^3(-2) \to 0. \tag{B.5}$$

Vanishing of the cohomology of $\mathcal{T}\mathbf{CP}^3(-2)$ now follows from the long exact sequence of cohomology groups derived from (B.5), together with the standard fact that $H^i(\mathbf{CP}^n, \mathcal{O}(-j)) = 0$ for all i and $0 < j < n+1$. From absence of zero modes of the Dirac operator on the fibers, it follows that the index bundle is trivial and hence that $I(\mathcal{T}) = 0$.

We now turn to $I(\wedge^2\mathcal{T})$. This case is more delicate, as the index bundle is not trivial. An argument like the one surrounding (B.2) lets us replace $\wedge^2\mathcal{T}$ by $\wedge^2\mathcal{T}\mathbf{CP}^3$. (In fact, for index purposes $\wedge^2\mathcal{T} = \wedge^2\mathcal{T}\mathbf{CP}^2 \oplus \mathcal{T}\mathbf{CP}^2 \otimes \mathcal{T}\mathbf{CP}^3 \oplus \wedge^2\mathcal{T}\mathbf{CP}^3$. Here, $\wedge^2\mathcal{T}\mathbf{CP}^2$ can be dropped as it is a pullback from the base, and $\mathcal{T}\mathbf{CP}^2 \otimes \mathcal{T}\mathbf{CP}^3$ can be dropped as its restriction to each fiber is isomorphic to $\mathcal{T}\mathbf{CP}^3$, which as we have just seen has zero cohomology and zero index bundle.) To analyze the index with values in $\wedge^2\mathcal{T}\mathbf{CP}^3$, first note that, in the last paragraph, just for computing the Dirac index with values in $\mathcal{T}\mathbf{CP}^3$ (as opposed to computing all of the cohomology groups of $\mathcal{T}\mathbf{CP}^3(-2)$, as we actually did), we could have assumed that the exact sequence in (B.3) splits, leading to the K-theory statement $\mathcal{T}\mathbf{CP}^3 = \mathcal{O}(1)^4 - \mathcal{O}$. Likewise, we have a K-theory statement $\wedge^2\mathcal{T}\mathbf{CP}^3 = \mathcal{O}(2)^6 - \mathcal{O}(1)^4$. After twisting by $K^{1/2} = \mathcal{O}(-2)$, we have

$$\wedge^2\mathcal{T}\mathbf{CP}^3(-2) = \mathcal{O}^6 - \mathcal{O}(-1)^4. \tag{B.6}$$

The index bundle of the Dirac operator with values in $\wedge^2\mathcal{T}\mathbf{CP}^3$ is the alternating sum of the cohomology groups of $\wedge^2\mathcal{T}\mathbf{CP}^3(-2)$. For computing the index bundle, we can replace this by $\mathcal{O}^6 - \mathcal{O}(-1)^4$. The only nonvanishing cohomology group of this bundle is $H^0(\mathbf{CP}^3, \mathcal{O}^6) = \mathbf{C}^6$.

The index bundle W of $\wedge^2\mathcal{T}\mathbf{CP}^3$ is thus a rank six complex vector bundle over $\mathbf{CP}^2$. To identify this bundle, we need to repeat the analysis in the last paragraph more precisely, to describe the dependence of the cohomology on the x_i. First of all, as (x_1, x_2, x_3) varies, the solution space of (B.1) varies as the bundle $M = U \oplus \mathcal{O} \oplus \mathcal{O}$ over $\mathbf{CP}^2$, where

$U = \mathcal{T}^*\mathbf{CP}^2(1)$. (A triple y_1, y_2, y_3 obeying (B.1) determines a differential form $\sum_i y_i\, dx_i$ of degree 1 on $\mathbf{CP}^2$; in the description of M, the summands $\mathcal{O}$ come from y_4 and y_5.) Hence, we can regard the $\mathbf{CP}^3$ fiber of $V_{1,1} \to \mathbf{CP}^2$ as $\mathbf{P}M$, the projectivization of M. In (B.3), we can regard $\mathcal{O}(1)^4$ as $M(1)$, and in (B.6), $\mathcal{O}^6$ is $\wedge^2 M \otimes \det(M)^{-1/2}$ (where the last factor will be explained in a moment). So the fiber of the index bundle W is really

$$H^0(\mathbf{P}M, \wedge^2\mathcal{T}\mathbf{P}M \otimes K^{1/2}(\mathbf{P}M)) = \wedge^2 M \otimes \det\, M^{-1/2}. \tag{B.7}$$

(The slightly delicate factor of det $M^{-1/2}$ on the right hand side can be explained as follows. The left hand side of (B.7) is manifestly invariant under the action of $\mathbf{C}^*$ on M; hence $\mathbf{C}^*$ must act trivially on the right hand side, which is so precisely if we include the given power of det M.) Now, with $M = U \oplus \mathcal{O} \oplus \mathcal{O}$, we find that $\wedge^2 M \otimes (\det M)^{-1/2}$ is

$$W = (\det U)^{1/2} \oplus (\det U)^{-1/2} \oplus 2U \otimes (\det U)^{-1/2}. \tag{B.8}$$

(Note that $\mathbf{CP}^2$ is not a spin manifold, but $(\det U)^{1/2}$ is a Spin^c structure on $\mathbf{CP}^2$. This is why the index bundle W is $(\det U)^{1/2}$ tensored with a conventional vector bundle.) Two copies of $U \otimes (\det U)^{-1/2}$ do not contribute to the mod 2 reduction of the index. As the index in four dimensions with values in a bundle E is invariant under complex conjugation of E, the index with values in $(\det U)^{1/2}$ equals that with values in $(\det U)^{-1/2}$, so these contributions to the mod 2 reduction of the index cancel also.

Appendix C. The Atiyah – Hirzebruch Spectral Sequence

The Atiyah – Hirzebruch spectral sequence (AHSS) is a systematic algebraic algorithm relating K-theory to integral cohomology. In this appendix we will give an elementary account of this formalism, explaining the construction of [23] in more familiar physical terms. In order to simplify the presentation we will focus on the relation between K^0 and even cohomology classes. The extension to K^1 and odd cohomology classes is straightforward.

Given a manifold X, we can study it's topology by introducing a triangulation that makes X look as a collection of simplexes glued together along their boundaries. Then we can determine the homology or the cohomology of X in terms of the gluing data using simple combinatorics. In order to do this in a systematic way, it is often convenient to think of X as a superposition of finitely many strata X^p, each stratum consisting of all simplicial cells of dimension p. X^p is called the p-skeleton of X.

When studying the K-theory of X, the stratification of X by skeletons induces a natural filtration of $K(X)$. We simply define $K^0_p(X)$ to be the subset of $K^0(X)$ consisting of classes which are trivial on the $(p-1)$-skeleton. In the present approach, it is more convenient to think of K-theory classes as D-branes on X. For concreteness, we assume the dimension N of X to be even. A $D(2p-1)$-brane wraps a $2p$-submanifold of X which is Poincaré dual to a cohomology class in $H^{N-2p}(X, \mathbf{Z})$. Such classes are supported on the $(N-2p)$-skeleton and cannot be detected on lower skeletons. Since $K^0_N(X)$ consists of classes trivial on the $(N-1)$ skeleton, it follows that the worldvolume of the corresponding D-brane states must be pointlike. These are D-instantons in IIB. Similarly, $K^0_{N-1}(X)$ consists of classes which are trivial on the $(N-2)$ skeleton. Since there are no stable even branes in IIB, it follows that $K^0_{N-1}(X) = K^0_N(X)$. Next, $K^0_{N-2}(X)$ parameterizes objects wrapping submanifolds of dimension two, i.e. $D1$-branes and so on. The complexity of $K^0_{N-2p}(X)$ increases as we increase p since a $D(2p-1)$-brane can have induced lower D-brane charges on it's world volume. We have accordingly the following sequence of inclusions

$$K^0_N(X) \subseteq K^0_{N-1}(X) \subseteq \cdots \subseteq K^0_0(X) = K^0(X). \tag{C.1}$$

Note that not all these inclusions are strict; in fact $K^0_{N-2p} = K^0_{N-2p-1}$ for all p. There is a similar filtration on $K^1(X)$, which can be described in terms of D-branes in IIA.

The main idea of the AHSS is that although $K^0_p(X)$ are complicated objects, it may be easier to determine the so-called "successive quotients"

$$K^0_p(X)/K^0_{p+1}(X). \tag{C.2}$$

Physically, this means that we are trying to understand D-brane charges starting with the lowest charges – such D-instantons in IIB – and working our way towards higher charges. For example D-instantons are pointlike on X, so the structure of $K^0_N(X)$ is very simple. D-instantons can dissolve in $D1$-branes forming bound states. These are classified by $K^0_{N-2}(X)$. Since we have already understood D-instantons, one might think that at the next step it suffices to focus only on $D1$-branes, regardless of their lower $D(-1)$-charge. As explained below, this is not quite true. If we ignore $D(-1)$-charges, $D1$-branes are classified by $K^0_{N-2}(X)/K^0_{N-1}(X)$. Proceeding similarly at each stage, we construct the "associated graded"

$$\mathrm{Gr}K^0(X) = \oplus_p K^0_p(X)/K^0_{p+1}(X). \tag{C.3}$$

Although this simplifies the computation of $K^0(X)$, some information is lost in the process. More precisely, $K^0(X)$ is not uniquely defined by the associated graded (C.3). When we try to construct $K^0(X)$, given $\mathrm{Gr}K^0(X)$, we have to determine a finite number of extensions of the form

$$0 \to K^0_{p+1}(X) \to K^0_p \to K^0_p(X)/K^0_{p+1}(X) \to 0, \tag{C.4}$$

which may be ambiguous. For example, if X is the real projective plane $\mathbf{RP}^5$, with the stratification given by the linear subspaces $X^{5-i} = \mathbf{RP}^i$, $0 \leq i \leq 5$, the associated graded is

$$\mathrm{Gr}K^0(\mathbf{RP}^5) = \mathbf{Z} \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2. \tag{C.5}$$

$K^0_1(\mathbf{RP}^5)$ is determined by the extension

$$0 \to \mathbf{Z}_2 \to K^0_1(\mathbf{RP}^5) \to \mathbf{Z}_2 \to 0 \tag{C.6}$$

which can be either $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ or $\mathbf{Z}_4$. Such problems can be solved by a more careful study of $K^0(X)$. In the present case, the solution is [40]

$$K^0_1(\mathbf{RP}^5) = \mathbf{Z}_4. \tag{C.7}$$

On the contrary, the next extension

$$0 \to \mathbf{Z}_4 \to K^0_0(\mathbf{RP}^5) \to \mathbf{Z} \to 0 \tag{C.8}$$

admits only the trivial solution, and the final result is

$$K^0(\mathbf{RP}^5) = \mathbf{Z} \oplus \mathbf{Z}_4. \tag{C.9}$$

The advantage of the associated graded (C.3) is that it can be determined by successive approximations. Recall that X is made of finitely many skeletons X^p, each skeleton consisting of finitely many simplicial cells σ^p_i. Each simplicial cell σ^p_i is topologically equivalent to a p-ball B^p. The boundary $\dot{\sigma}^p_i$ of σ^p_i consists of $(p+1)$ faces which are $(p-1)$-simplexes themselves belonging to the $(p-1)$-skeleton of X. The simplest object that can be formed out of this local data is

$$E^p_1 = K^0(X^p, X^{p-1}) = \oplus_i K^0(\sigma^p_i, \dot{\sigma}^p_i). \tag{C.10}$$

E^p_1 parameterizes K-theory classes defined on the p-skeleton which are trivial on the $(p-1)$-skeleton. Note however that at this stage we do not know if these classes can be lifted to full

K-theory classes on X. That is why E_1^p can be thought as a zeroth order approximation to $K_p^0(X)/K_{p+1}^0(X)$. In mathematical terms, E_1^p is called the first term of the spectral sequence.

Before moving on, let us rewrite (C.10) in a more familiar form. K-theory classes on σ_i^p which are trivial on the boundary can be identified with classes on the p-sphere S^p by collapsing $\dot{\sigma}_i^p$. More precisely, $K^0(\sigma_i^p, \dot{\sigma}_i^p)$ can be identified with the reduced K^0-theory of a sphere S^p, which is isomorphic to K^p of a point by Bott periodicity. Therefore the first term E_1^p can be identified with singular p-cochains on X with values in $K^p(x_0)$ (x_0 is an arbitrary base point of X)

$$E_1^p = C^p(X; K^p(x_0)) = \begin{cases} C^p(X, \mathbf{Z}), & p \text{ even} \\ 0, & p \text{ odd.} \end{cases} \tag{C.11}$$

Higher AHSS approximations involve a systematic refinement of (C.10). We want to characterize the K-theory classes in E_1^p which can be lifted to X. This question can be answered inductively, by first determining the classes which can be lifted to the $(p+1)$-skeleton. These will form a second term E_2^p, which can be further refined by restricting to classes which can be lifted to the $(p+2)$-skeleton and so on. The power of this approach resides in the fact that at each step, one can define a differential

$$d_r^p : E_r^p \rightarrow E_r^{p+r}, \qquad d_r^p \circ d_r^{p-r} = 0 \tag{C.12}$$

such that E_{r+1}^p is the cohomology of d_r

$$E_{r+1}^p = \mathrm{Ker}(d_r^p)/\mathrm{Im}(d_r^{p-r}). \tag{C.13}$$

The spectral sequence consists of the collection (E_r, d_r) obtained by summing over all p. In practice, after finitely many steps, this sequence becomes stationary and we obtain the successive quotient (C.2). The spectral sequence is said to converge to the associated graded (C.3).

This construction can be understood in terms of D-branes as well. As discussed in detail in sections 5.1.– 5.2., a D-brane cannot wrap a submanifold Q of X unless the Poincaré dual class b can be lifted to K-theory. In the process of constructing the associated K-theory class, one has to extend the tachyon condensate (5.5) as a unitary map between two bundles over the entire manifold X. The AHSS is simply an algorithm for keeping track of the possible obstructions. At each stage we keep only states for which T can be

extended a finite number of steps. At the same time, we have to mod out by unstable states, for which the extension is trivial. The whole procedure is reminiscent of a refined BRST quantization scheme, consisting of a sequence of BRST operators d_r, each acting on the space of physical states of the previous one. The true physical space is obtained by taking the cohomology of all operators.

Let us work out d_1^p. The derivation is essentially identical to the extension of the tachyon field in section 5.2. We assume p even in order to have a nontrivial E_1^p. Let σ^{p+1} be an arbitrary $(p+1)$-simplex with faces σ_i^p. Suppose we have K-theory classes x_i in $K^0(\sigma_i^p, \dot{\sigma}_i^p)$. As we saw in section 5.2, K^1classes are classified by homotopy classes of maps to the infinite unitary group U. Here we have K^0 classes which can be similarly classified by maps to the loop space[19] of U, ΩU. So we can think of x_i as maps $f : \sigma_i^p \to \Omega U$ mapping the boundary to a point, or equivalently as maps $f_i : S^p \to \Omega U$. Since the σ_i^p are the faces of σ^{p+1}, the maps f_i can be glued along the boundaries, resulting in a map $f : \dot{\sigma}^{p+1} \to \Omega U$. With the orientations properly taken into account, the homotopy class of this map in $\pi_p(\Omega U) = \mathbf{Z}$ is

$$[f] = \sum_i (-1)^i [f_i]. \tag{C.14}$$

The map f can be extended to the interior of σ^{p+1} if and only if $[f]$ is trivial in $\pi_p(\Omega U)$. Therefore $[f]$ is the obstruction to extending $\{x_i\}$ to the $(p+1)$-skeleton. By assigning such an $[f]$ to all $(p+1)$-simplexes, we obtain a map from the p-cochains of X to the $(p+1)$-cochains. This is the first AHSS differential

$$d_1^p : C^p(X; K^p(x_0)) \to C^{p+1}(X; K^p(x_0)). \tag{C.15}$$

According to (C.14), d_1^p is precisely the standard simplicial coboundary operator.

Following the steps outlined above, the second term in the AHSS is therefore

$$E_2^p = H^p(X; K^p(x_0)) = \begin{cases} H^p(X; \mathbf{Z}), & p \text{ even} \\ 0, & p \text{ odd.} \end{cases} \tag{C.16}$$

For K^1, we obtain an analogous formula, with p even replaced by p odd. This explains the formulae (5.1) and (5.2) in the main text. E_2^p can be regarded as a first order approximation to $K_p^0(X)/K_{p+1}^0(X)$.

[19] The only property of ΩU needed in the following is that $\pi_i(\Omega U) = \pi_{i+1}(U)$.

We can continue this process in a similar manner (see the discussion of the extension of T in section 5.2.) The next differential is trivial since $\pi_{p+1}(\Omega U)$ is trivial if p is even. This argument generalizes to all even differentials, showing that

$$d^p_{2r} = 0. \tag{C.17}$$

So the next nontrivial obstruction is encountered when extending over the $(p+3)$-skeleton and it takes values in $\pi_{p+2}(\Omega U)$. The corresponding AHSS differential is a "cohomology operation"

$$d^p_3 : H^p(X;\mathbf{Z}) \to H^{p+3}(X;\mathbf{Z}). \tag{C.18}$$

Apparently there is no simple derivation of d^p_3, although the arguments in section 5.1. suggest that

$$d^p_3 = Sq^3. \tag{C.19}$$

This is indeed the correct answer [23].

Appendix D. Spin Ten-Manifolds with $W_7 \neq 0$

In section 6.1 we found that M-theory on a spin manifold of the form $X \times S^1$ is inconsistent if $W_7(X) \neq 0$. This is an important constraint on the theory and we would like to know if spin ten-manifolds with $W_7(X) \neq 0$ exist. Unfortunately, there does not seem to exist an elementary example. However, the existence of such manifolds can be inferred from an abstract cobordism argument explained to us by Mike Hopkins. The main idea is to use the Pontryagin duality

$$H^7(X;\mathbf{Z}) \times H^3(X;\mathbf{Q}/\mathbf{Z}) \to \mathbf{Q}/\mathbf{Z}. \tag{D.1}$$

Regarding $\mathbf{Q}/\mathbf{Z}$ as a subgroup of $U(1)$, this can be related to the duality discussed in section 4.2 (see formula (4.30).) We have similarly a short exact sequence

$$0 \to \mathbf{Z} \to \mathbf{Q} \to \mathbf{Q}/\mathbf{Z} \to 0 \tag{D.2}$$

and a Bockstein map $\beta : H^k(X;\mathbf{Q}/\mathbf{Z}) \to H^{k+1}(X;\mathbf{Z})$. Since the group $\mathbf{Z}_2$ can be embedded in $\mathbf{Q}/\mathbf{Z}$, we can regard the Stiefel-Whitney classes $w_k(X)$ as elements of $H^k(X;\mathbf{Q}/\mathbf{Z})$. With this understanding we have

$$W_7(X) = \beta(w_6(X)). \tag{D.3}$$

The pairing (D.1) is nondegenerate, hence $W_7(X)$ is nonzero if and only if there exists an element $\xi \in H^3(X; \mathbf{Q}/\mathbf{Z})$ such that

$$I(X, \xi) = \int_X \xi \cup W_7(X) \neq 0. \tag{D.4}$$

Therefore, it suffices to establish the existence of a pair (X, ξ) such that $I(X, \xi) \neq 0$. Note that $I(X, \xi)$ is a cobordism invariant of the pair (X, ξ) in the sense explained in section 3.2. The class ξ is classified by a map $f : X \to K(\mathbf{Q}/\mathbf{Z}, 3)$ and $I(X, \xi) \in \mathrm{Hom}\left(\widetilde{\Omega}_{10}^{spin}(K(\mathbf{Q}/\mathbf{Z}, 3)), \mathbf{Q}/\mathbf{Z}\right)$. Now, the invariant (D.4) can be rewritten

$$I(X, \xi) = \int_X \beta(\xi) \cup w_6(X), \tag{D.5}$$

where we have to use the pairing between $H^4(X; \mathbf{Z})$ and $H^6(X; \mathbf{Q}/\mathbf{Z})$. Note that this is in fact a familiar cobordism invariant encountered in section 3.2. Setting $a = \beta(\xi) \in H^4(X; \mathbf{Z})$, we have $I(X, \xi) = v(a)$ as defined in equation (3.16). This is an invariant of the group $\widetilde{\Omega}_{10}^{spin}(K(\mathbf{Z}, 4))$. Moreover, we can find a bordism class (Y, a) in $\widetilde{\Omega}_{10}^{spin}(K(\mathbf{Z}, 4))$ such that

$$\int_Y a \cup w_6(Y) = 1. \tag{D.6}$$

For example, pick Y to be the degree $(1,1)$ hypersurface $V_{1,1}$ introduced in section 3.3 (below (3.28)), and $a = \lambda(V_{1,1})$.[20] Note that $a = g^*(u)$ where $g : Y \to K(\mathbf{Z}, 4)$ is a continuous map and $u \in H^4(K(\mathbf{Z}, 4); \mathbf{Z})$ is the standard generator.

Given the existence of a such a pair (Y, g), in order to find a pair (X, f) as above it suffices to prove that the bordism groups $\widetilde{\Omega}_{10}^{spin}(K(\mathbf{Q}/\mathbf{Z}, 3))$ and $\widetilde{\Omega}_{10}^{spin}(K(\mathbf{Z}, 4))$ are isomorphic. Note that the exact sequence (D.2) induces a canonical map $\pi : K(\mathbf{Q}/\mathbf{Z}, 3) \to K(\mathbf{Z}, 4)$. Let C_π denote the mapping cone of π so that we have a sequence of maps of the form

$$K(\mathbf{Q}/\mathbf{Z}, 3) \xrightarrow{\pi} K(\mathbf{Z}, 4) \to C_\pi. \tag{D.7}$$

This induces a long exact sequence of bordism groups which reads in part

$$\cdots \widetilde{\Omega}_{11}^{spin}(C_\pi) \to \widetilde{\Omega}_{10}^{spin}(K(\mathbf{Q}/\mathbf{Z}, 3)) \xrightarrow{\pi_*} \widetilde{\Omega}_{10}^{spin}(K(\mathbf{Z}, 4)) \to \widetilde{\Omega}_{10}^{spin}(C_\pi) \cdots \tag{D.8}$$

[20] In fact we have to be slightly more careful here. The pair (Y, a) does not define an invariant of the reduced bordism group $\widetilde{\Omega}_{10}^{spin}(K(\mathbf{Z}, 4))$ since Y is not a boundary. We have to replace Y by a sum of two copies of $V_{1,1}$, one of them with reversed orientation. Then we take a to be $\lambda(V_{1,1})$ supported on one of the two components.

We conclude that $\pi_* : \widetilde{\Omega}_{10}^{spin}(K(\mathbf{Q}/\mathbf{Z}, 3)) \to \widetilde{\Omega}_{10}^{spin}(K(\mathbf{Z}, 4))$ is an isomorphism if one can prove that the bordism groups $\widetilde{\Omega}_{10}^{spin}(C_\pi), \widetilde{\Omega}_{11}^{spin}(C_\pi)$ vanish.

This follows from the Atiyah-Hirzebruch spectral sequence for C_π. Note that by construction the integral homology of C_π is given by $H_*(C_\pi, \mathbf{Z}) \simeq H_*(K(\mathbf{Z}, 4)) \otimes_{\mathbf{Z}} \mathbf{Q}$. Using the universal coefficient theorem, one can easily establish that $H_*(K(\mathbf{Z}, 4)) \otimes_{\mathbf{Z}} \mathbf{Q} \simeq \mathbf{Q}[\xi]$, where ξ is a degree four generator. The second term in the AHSS for C_π is $E^2_{p,q} = \widetilde{H}_p\left(C_\pi, \Omega_q^{spin}\right)$. For $p + q \leq 12$, we obtain the following nontrivial terms

$$\begin{array}{c|ccc} 8 & & \mathbf{Q}^2 & \\ 4 & & \mathbf{Q} & \mathbf{Q} \\ 0 & & & \\ \hline & 0 & 4 & 8 \end{array}$$

Therefore $\widetilde{\Omega}_{11}^{spin}(C_\pi) = \widetilde{\Omega}_{10}^{spin}(C_\pi) = 0$ since all terms in degrees $p + q = 10, 11$ are zero, and we obtain the desired isomorphism.

Let us make this isomorphism more explicit. We will mainly exploit the surjectivity of π_*. Given the cobordism class of (Y, g) defined above, it follows that there must exist a class (X, f), $f : X \to K(\mathbf{Q}/\mathbf{Z}, 3)$ such that

$$(Y, g) = \pi_*(X, f). \tag{D.9}$$

The interpretation of this relation is quite elementary, according to section 3.2. Namely, it means that we can find a spin manifold Z with boundary $X - Y$ and a map $F : Z \to K(\mathbf{Z}, 4)$ such that F restricted to X is $\pi \circ f : X \to K(\mathbf{Z}, 4)$ and F restricted to Y is g. This shows that

$$\int_X (\pi \circ f)^*(u) \cup w_6(X) = \int_Y g^*(u) \cup w_6(Y) \tag{D.10}$$

since the expressions in question are cobordism invariants. We have $(\pi \circ f)^*(u) = f^*\pi^*(u)$ where $\pi^*(u)$ is the pull back of u to $H^4(K(\mathbf{Q}/\mathbf{Z}, 3); \mathbf{Z})$. This group consists entirely of elements of the form $\beta(\eta)$, with $\eta \in H^3(K(\mathbf{Q}/\mathbf{Z}, 3); \mathbf{Q}/\mathbf{Z})$. Therefore there exists such an element η so that $\pi^*(u) = \beta(\eta)$. The pull back $f^*(\eta)$ defines an element $\xi \in H^3(X, \mathbf{Q}/\mathbf{Z})$. Collecting all the facts, it follows that there must exist a pair (X, ξ) such that

$$I(X, \xi) = \int_X \xi \cup W_7(X) = \int_Y a \cup w_6(Y) = 1. \tag{D.11}$$

This proves the claim.

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